Interplay of type I and type II seesaw contributions to neutrino mass

Evgeny Kh. Akhmedov[*]

Department of Theoretical Physics, Royal Institute of Technology,
AlbaNova University Center, SE-106 91 Stockholm, Sweden
E-mail: `akhmedov@ictp.trieste.it`

Michele Frigerio

Service de Physique Théorique, CEA-Saclay,
91191 Gif-sur-Yvette Cedex, France
E-mail: `frigerio@spht.saclay.cea.fr`

ABSTRACT: Type I and type II seesaw contributions to the mass matrix of light neutrinos are inherently related if left-right symmetry is realized at high energy scales. We investigate implications of such a relation for the interpretation of neutrino data. We proved recently that the left-right symmetric seesaw equation has eight solutions, related by a duality property, for the mass matrix of right-handed neutrinos M_R. In this paper the eight allowed structures of M_R are reconstructed analytically and analyzed numerically in a bottom-up approach. We study the dependence of right-handed neutrino masses on the mass spectrum of light neutrinos, mixing angle θ_{13}, leptonic CP violation, scale of left-right symmetry breaking and on the hierarchy in neutrino Yukawa couplings. The structure of the seesaw formula in several specific SO(10) models is explored in the light of the duality. The outcome of leptogenesis may depend crucially on the choice among the allowed structures of M_R and on the level crossing between right-handed neutrino masses.

KEYWORDS: Beyond Standard Model, Neutrino Physics, GUT.

---

[*]On leave from the National Research Centre Kurchatov Institute, Moscow, Russia

Contents

1. Introduction

Experiments with solar, atmospheric, accelerator and reactor neutrinos have given an un-ambiguous evidence for neutrino oscillations and therefore for neutrino mass. This implies a physics beyond the standard model (SM) of particle physics, since neutrinos are strictly massless in that model. At the same time, neutrino mass is different from zero in almost

every extension of the SM. Moreover, many models contain more than one source of neutrino mass. A crucial issue is therefore the ability to discriminate, within a given theoretical framework, among these sources using neutrino data as well as other neutrino-related experimental input.

If lepton number is not conserved, neutrinos are Majorana particles and their mass originates from the dimension five operator $LL\phi\phi$ via the electroweak symmetry breaking ($L \equiv (\nu_L\ l_L)^T$ and $\phi \equiv (\phi^+\ \phi^0)^T$ are the SM lepton and Higgs isodoublets, respectively). New physics can contribute to this operator either at tree or at loop level. We consider here the first possibility; the super-heavy particles exchanged at tree level can then be either isosinglet fermions N_R, that is, right-handed (RH) neutrinos [1], or isotriplet scalars Δ_L [2].[1] The two cases are known under the names of type I and type II seesaw mechanism, respectively. Besides providing a very natural explanation of the smallness of the neutrino mass, the seesaw mechanism has a simple and elegant built-in mechanism for generating the observed baryon asymmetry of the universe – baryogenesis via leptogenesis [4]. If the masses of N_R and Δ_L are much larger than the electroweak scale, then neutrino experiments can only probe the couplings of the effective operator. Therefore in general the data cannot tell type I from type II seesaw contribution to the masses of light neutrinos. Nonetheless, in several (possibly unified) models of fermion masses and mixing, both type I and type II terms are naturally present, and the knowledge of their relative size is crucial for achieving predictivity.[2]

In this paper we study in detail the interplay of type I and type II seesaw contributions to neutrino mass. We consider a theoretically well motivated connection between these two contributions, which stems from the proportionality of the Majorana mass terms of left-handed and RH neutrinos. Let the Lagrangian of the theory contain the following term:

$$\mathcal{L}_M = -\frac{f}{2}(\nu_L^T C \nu_L \Delta_L^0 + N_L^{cT} C N_L^c \Delta_R^0) + \text{h.c.}\,, \tag{1.1}$$

where $C = i\gamma_2\gamma_0$ is the charge conjugation matrix, $N_L^c \equiv (N_R)^c$ and $\Delta_{L,R}^0$ are neutral scalar fields. In the case of more than one generation, ν_L and N_L^c carry a flavor index and f is a symmetric matrix of couplings. When $\Delta_{L,R}^0$ develop non-zero vacuum expectation values (VEVs) $v_{L,R}$, two Majorana mass terms, proportional to each other, are generated, one for left-handed and one for RH neutrinos. In general, a Dirac-type Yukawa coupling term is also present in the theory:

$$\mathcal{L}_D = -y\nu_L^T C N_L^c \phi^0 + \text{h.c.}\,, \tag{1.2}$$

where the neutral scalar ϕ^0 has a VEV v. When $v_R \gg v$, the effective Majorana mass matrix for light left-handed neutrinos takes the form

$$m_\nu \simeq v_L f - \frac{v^2}{v_R} y f^{-1} y^T\,. \tag{1.3}$$

---

[1]We will not consider the unique remaining possibility, that is, isotriplet fermions [3], which are more difficult to embed in minimal extensions of the SM.

[2]Even in very constrained scenarios, it is far from trivial to control the relative size of the contributions of the two seesaw types. This problem in the minimal SUSY SO(10) model has been extensively discussed in [5, 6] (for a different model see, e.g., [7]).

The first (second) term on the r.h.s. of this equation is known as type II (type I) contribution to the light neutrino mass.[3] The same coupling f enters in both contributions because of the assumption made in eq. (1.1), which follows from a discrete left-right symmetry of the underlying theory.

In fact, the seesaw formula takes the form given in eq. (1.3) in models with left-right (LR) symmetry above some energy scale larger than the electroweak scale [8, 9]. By this we mean that the gauge group contains (or coincides with) $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ *and* a discrete symmetry guarantees that the $SU(2)_L$ and $SU(2)_R$ gauge couplings are asymptotically equal.[4] Such models incorporate naturally RH neutrinos (as well as isotriplet scalars) and explain the maximal parity violation of low-energy weak interactions as a spontaneous symmetry breaking phenomenon. Note also that LR models can be easily made supersymmetric, and the exact R-parity at low energies can be obtained through the spontaneous breaking of $U(1)_{B-L}$ [11].

In this paper we perform a thorough phenomenological analysis of the seesaw formula (1.3). A duality property of this formula, which will be at the basis of our analysis, was identified in a recent paper by the authors [12]. The implications of such duality in a specific SO(10) model are studied in [13].

The paper is organized as follows. In section 2 we discuss the duality property of the left-right symmetric seesaw. We also give the motivation for the reconstruction of the matrix f (and therefore of the mass matrix of heavy RH neutrinos) from the seesaw formula (1.3) and study the multiplicity of the solutions. In section 3 we develop a method of exact analytic reconstruction of f in the cases of one, two and three lepton generations. In section 4 a number of numerical examples, illustrating our analytic results, are given and the conditions for the existence of light RH neutrinos are discussed. In section 5 we discuss the seesaw duality in specific models with LR symmetry, including several SO(10) models. In section 6 we briefly discuss some further issues pertaining to our analysis – stability of the results with respect to the renormalization group evolution effects and baryogenesis through leptogenesis. Discussion and summary are given in section 7. The appendices contain further analytic treatment of relevant topics: the reconstruction of f in the case of antisymmetric y (appendix A), the estimate of the relative size of type I and II seesaw contributions to m_ν (appendix B), and the generalization of the duality and the reconstruction of f in the case of a different realization of the discrete LR symmetry (appendix C).

2. Bottom-up approach to left-right symmetric seesaw

The LR symmetric seesaw equation (1.3) relates three vacuum expectation values (v, v_L and v_R) and three $n \times n$ matrices (m_ν, y and f), where n is the number of lepton generations.

---

[3]Sometimes in the literature the mechanism leading to eq. (1.3) is called type II seesaw. We prefer the terminology where it is called type I+II seesaw, whereas the term "type II seesaw" is reserved for the situation when the Higgs triplet Δ_L is the sole source of neutrino mass.

[4]Classic extensions of this minimal LR symmetric setting are provided by the Pati-Salam model, based on $SU(4)_c \times SU(2)_L \times SU(2)_R \equiv SU_{422}$ [8], and by the unified models based on SO(10) [10].

The matrix f is not directly constrained by any experiment. Moreover, being a Majorana-type Yukawa coupling matrix, f has no analogy with the only Yukawa coupling matrices that can be presently accessed experimentally (y_u, y_d, y_e), which are all of Dirac type.

It is therefore natural to employ the seesaw formula for reconstructing the matrix f, taking the quantities m_ν, y, v, $v_{L,R}$ as input parameters. The purpose is to provide an insight into the underlying theory at the seesaw scale, which is deeply characterized by the structure of f and which is not accessible to direct experimental studies. Notice that eq. (1.3) is a non-linear $n \times n$ matrix equation for f and solving it in the case $n > 1$ is highly non-trivial.

In this section we first review the experimental and theoretical constraints on the input parameters. Then we discuss the duality property of the LR symmetric seesaw formula (1.3) and the multiplicity of its solutions for the matrix f.

2.1 Parameter space

Let us examine the possible values of the various quantities involved in eq. (1.3).

The neutrino mass matrix m_ν may be completely determined, at least in in principle, from low energy experiments. In terms of the mass eigenvalues m_i ($i = 1, 2, 3$) and leptonic mixing matrix U, one has

$$m_\nu = U^* diag(m_1,\, m_2,\, m_3)\, U^\dagger \,, \tag{2.1}$$

where U depends on three mixing angles θ_{12}, θ_{23} and θ_{13}, one Dirac-type CP-violating phase δ and two Majorana-type CP-violating phases ρ and σ. From global fits of low energy experiments [14, 15] one finds

$$\Delta m_{\rm sol}^2 \equiv m_2^2 - m_1^2 = (7.9^{+1.0}_{-0.8}) \times 10^{-5} \; {\rm eV}^2 \,, \qquad \Delta m_{\rm atm}^2 \equiv |m_3^2 - m_1^2| = (2.6 \pm 0.6) \times 10^{-3} \; {\rm eV}^2 \,, \tag{2.2}$$

$$\theta_{12} = (33^{+6}_{-4})^\circ \,, \qquad \theta_{23} = (45 \pm 10)^\circ \,, \qquad \theta_{13} \lesssim 12^\circ \,, \tag{2.3}$$

where the best fit values and 3σ intervals are taken from the last update of [14] (v5). Cosmology yields the most stringent upper limit on the sum of the neutrino masses, $m_1 + m_2 + m_3 \lesssim 0.4(0.7)\,{\rm eV}$ at 99.9% C.L. with(without) including Lyman-α data in the fit [16]. The values of the CP-violating phases are completely unknown at present.

As far as the vacuum expectation values are concerned, we choose the convention of real and positive v, v_L and v_R (this can always be achieved by redefining the phases of ν_L, N_L^c and y in eqs. (1.1) and (1.2)). The electroweak symmetry breaking parameter $v \approx 174\,{\rm GeV}$ is accurately known, while only weak constraints are available for the triplet VEVs $v_{L,R}$. No lower bound on v_L exists, and actually conditions are known under which Δ_L does not acquire any induced VEV. In contrast to this, v_L is bounded from above by its contribution to the ρ-parameter ($\Delta\rho \approx -2v_L^2/v^2$), so that precision electroweak measurements imply $v_L \lesssim {\rm GeV}$) [17]. Since we do not know the scale for the onset of LR symmetry, there is essentially no upper bound on v_R; we will be assuming in the following $v_R < M_{\rm Pl}$. The value of v_R is bounded from below by the non-observation of RH weak currents ($v_R \gtrsim {\rm TeV}$) [17]. It is useful to translate these bounds into bounds on the parameter $x \equiv v_L v_R / v^2$. If all

dimensionless couplings in the scalar potential are of order one, its natural value is $x \sim 1$. However, the scalar sector may well be very complicated and the potential may depend on various mass scales which are not constrained *a priori*. Therefore, x should be considered a free parameter in a model independent analysis. Using the above experimental constraints, one finds $0 \leq x \lesssim 10^{14}$.

The Dirac-type neutrino Yukawa coupling matrix y is not directly measurable, at least at present. Nonetheless, in LR-symmetric models and in their embedding in models with partial or grand unification, y is usually related to known quark and/or charged lepton Yukawa couplings. A detailed discussion of the structure of y in some of these models will be given in section 5. Here we constrain ourselves to a few examples. In the minimal LR symmetric model, in the supersymmetric case one has $y = \tan\beta y_e$, where y_e is the Yukawa coupling matrix of charged leptons and $\tan\beta$ is the usual ratio of VEVs. In the minimal Pati-Salam model, one obtains instead $y = y_u$, where y_u is the Yukawa coupling matrix of the up-type quarks. This relation also holds in the SO(10) model where the only Higgs multiplet contributing to the fermion masses is in the fundamental representation $\mathbf{10}_H$. More complicated (and realistic) examples will be discussed in section 5.

One should bear in mind, however, that the nature of the high energy theory is essentially unknown and therefore the Yukawa coupling matrix of neutrinos may in fact be very different from those of charged fermions. In particular, the neutrino flavor sector may have radically different symmetries. As an interesting example, one may consider $y = diag(a, b, b)$ [18], which is motivated by the maximal or nearly maximal 2-3 mixing observed in the atmospheric neutrino oscillations. It should be also noted that, in the presence of low energy supersymmetry, lepton flavor violating processes can be directly sensitive to the neutrino Yukawa couplings. For example, in constrained mSUGRA seesaw models one may need $y_{ij} \ll 1$ for some i, j, in order to suppress processes like $\tau \to \mu\gamma$ [19].

This brief survey illustrates that although certain choices of y may be well motivated, this matrix is in fact model dependent. In the following we will keep the form of y as general as possible, so as to maintain the generality of our results within the chosen LR symmetric framework.

2.2 Seesaw duality

Before discussing the general case, it will be useful to find the matrix f approximately in the limits when either of the two contributions to m_ν in eq. (1.3) dominates (these limits are well defined as long as all the matrix elements in m_ν are dominated by the same type of seesaw). In the case of dominant type I seesaw, one obtains

$$
\begin{aligned}
f_I = &-\frac{1}{x} y^T m^{-1} y + \frac{1}{x^2} (y m^{-1} y)^T m^{-1} (y m^{-1} y) \\
&- \frac{1}{x^3} (m^{-1} y m^{-1} y)^T (y^T m^{-1} y + y m^{-1} y^T)(m^{-1} y m^{-1} y) + \dots,
\end{aligned}
\tag{2.4}
$$

where we used the notation

$$
m \equiv \frac{m_\nu}{v_L}, \qquad x \equiv \frac{v_L v_R}{v^2}.
\tag{2.5}
$$

In the case of dominant type II seesaw, one finds

$$f_{II} = m + \frac{1}{x}ym^{-1}y^T - \frac{1}{x^2}(ym^{-1}y)m^{-1}(ym^{-1}y)^T + \dots. \tag{2.6}$$

In this latter case $m_\nu \approx v_L f_{II}$, so that low energy neutrino data allow one to reconstruct directly the mass matrix of RH neutrinos $M_R \approx v_R f_{II}$. In particular, the spectrum of the heavy neutrinos coincides (up to an overall factor) with that of light neutrinos [20].

In general, there is no *a priori* reason to expect that one type of seesaw dominates. If both contributions to m_ν are comparable, the solution of eq. (1.3) cannot be obtained expanding around the purely type I or II solution and a different approach is needed.

The LR symmetric type I+II seesaw formula in eq. (1.3) has the following intriguing duality property. Suppose that a matrix f solves this equation. Then one can verify that

$$\hat{f} \equiv m - f = -\frac{1}{x}yf^{-1}y^T \tag{2.7}$$

is also a solution, provided that the matrix y is invertible and symmetric (or antisymmetric). These conditions on y turn out to be both necessary and sufficient for $\hat{f}$ to solve eq. (1.3). In particular, they guarantee that the matrix $\hat{f}$ is invertible, as it must be in order to satisfy eq. (1.3). It is easy to see that the duality operation is closed, i.e. the dual of $\hat{f}$ coincides with f.

An important example of duality is provided by the solutions for f in the case of one seesaw type dominance, given in eqs. (2.4) and (2.6): if $y = \pm y^T$, one immediately sees that f_I and f_{II} are dual to each other. This allows us to establish an important result: if there is a solution of the seesaw equation with dominant type I seesaw, then there is also an alternative one, with dominant type II seesaw, and vice versa. More generally, we will show that the LR symmetric seesaw equation always has a pair of dual solutions which reduce to f_I and f_{II} when the input parameters satisfy certain conditions.

In the following we will focus on the case where y is symmetric. This is true in models with the discrete LR symmetry $\nu_L \leftrightarrow N_L^c$ (see eq. (1.2)), including the minimal LR symmetric model and SO(10) models where only the couplings to $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ Higgs multiplets contribute to y (for further details see section 5). The case of antisymmetric y will be discussed in appendix A. When y is symmetric, one can write $y = U y_d U^T$, where y_d is diagonal, real and positive and U is a unitary matrix. (We consider m_ν in the basis where the mass matrix of charged leptons is diagonal, so that U describes the mismatch between the left-handed rotations that diagonalize the Yukawa couplings of charged leptons and neutrinos.) As a consequence, eq. (1.3) can be rewritten as

$$U^\dagger m_\nu U^* = v_L(U^\dagger f U^*) - \frac{v^2}{v_R}y_d(U^\dagger f U^*)^{-1}y_d. \tag{2.8}$$

Therefore, one can always work (and we will) in the basis where y is diagonal, by redefining $(U^\dagger m_\nu U^*) \to m_\nu$ and $(U^\dagger f U^*) \to f$. If $U \approx \mathbb{1}$, the input from the low-energy neutrino data will still (approximately) determine the left-hand side of eq. (2.8). Note that U is the leptonic analogue of the CKM mixing matrix in the quark sector, where mixing is known to be small. Therefore the condition $U \approx \mathbb{1}$ may be motivated by a quark-lepton symmetry.

2.3 Multiplicity of solutions

We have found that, if a matrix f solves the seesaw equation (1.3), so does $\hat{f} \equiv m - f$, i.e. this equation does not have a unique solution. This is hardly surprising, as eq. (1.3) is a non-linear matrix equation for f, or, equivalently, a system of non-linear coupled equations for its elements f_{ij}. From the duality property of eq. (1.3) it immediately follows that the number of its solutions must be even. We shall now show that for n lepton generations the multiplicity of solutions is 2^n.[5]

Let us introduce the matrices $\tilde{m}$ and $\tilde{f}$ through the relations

$$m = \frac{1}{\sqrt{x}} y^{1/2} \, \tilde{m} \, y^{1/2} \,, \qquad\qquad f = \frac{1}{\sqrt{x}} \, y^{1/2} \tilde{f} \, y^{1/2} \,, \qquad (2.9)$$

where $y^{1/2}$ satisfies $(y^{1/2})^2 = y$. From eq. (1.3) with symmetric and invertible y one then obtains

$$\tilde{m} = \tilde{f} - \tilde{f}^{-1} \,. \qquad (2.10)$$

The duality we discussed above is especially clearly seen in this equation, where it corresponds to the invariance with respect to $\tilde{f} \leftrightarrow -\tilde{f}^{-1} \, (= \tilde{m} - \tilde{f})$. Multiplying eq. (2.10) by $\tilde{f}$ on the left or on the right, we find that $\tilde{f}$ satisfies

$$\tilde{f}^2 - \tilde{m}\tilde{f} - \mathbb{1} = 0 \qquad (2.11)$$

and that $[\tilde{m}, \tilde{f}] = 0$. The commutativity of $\tilde{m}$ and $\tilde{f}$ allows one to find a formal solution of eq. (2.11). Indeed, let us write $\tilde{f}$ in the form $\tilde{f} = \tilde{m}/2 + R$, where the matrix R is yet to be determined. Substituting this into eq. (2.11) in which the term $\tilde{m}\tilde{f}$ is rewritten as $(1/2)\{\tilde{m}, \tilde{f}\}$, one finds that the matrix R satisfies $R^2 = \frac{\tilde{m}^2}{4} + \mathbb{1}$, so that finally one obtains

$$\tilde{f} = \frac{\tilde{m}}{2} + \sqrt{\frac{\tilde{m}^2}{4} + \mathbb{1}} \,. \qquad (2.12)$$

Since any non-singular $n \times n$ matrix with non-degenerate eigenvalues has 2^n square roots [21], eq. (2.10) (and so eq. (1.3)) has 2^n solutions. Obviously, if R_0 is a square root of R^2, so is $-R_0$; therefore the solutions in eq. (2.12) form 2^{n-1} dual pairs with $\tilde{f} + \hat{\tilde{f}} = \tilde{m}$.

3. Analytic reconstruction of the coupling matrix f

In this section we will solve analytically the seesaw equation for f in the cases of one, two and three lepton generations. For one generation, the analysis is straightforward; for more than one lepton flavor, the effects of mixing complicate the reconstruction of f considerably and, for this purpose, we shall need to develop new algebraic techniques. We will provide general solutions for the case of symmetric y, where the seesaw duality occurs (the generalization to antisymmetric y is given in appendix A).

---

[5]This result was first obtained in [12], though in a less straightforward way. An alternative derivation can be found in [13].

We will also identify the criteria to quantify the dominance of one or the other type of seesaw. In the presence of mixing, this identification will turn out to be a subtle issue, whose details are given in appendix B.

A short account of the main results of this section was published in [12]. An alternative approach for the analytic reconstruction of f was developed in [13], for the case of symmetric y.

3.1 The case of one lepton generation

In this case m_ν, y and f are merely complex numbers. Eq. (1.3), being quadratic in f, has two solutions, which we denote $f_\pm$:

$$v_L f_\pm = \frac{1}{2}\left[m_\nu \pm \left(m_\nu^2 + \frac{4v^2 y^2 v_L}{v_R}\right)^{1/2}\right] = \frac{m_\nu}{2}\left[1 \pm (1+d)^{1/2}\right], \tag{3.1}$$

where

$$d \equiv \frac{v_L}{v_R}\frac{4v^2 y^2}{m_\nu^2}. \tag{3.2}$$

Obviously, the solutions $f_\pm$ are dual to each other: $m = f_+ + f_-$.

At this point it is useful to state our convention for the assignment of the complex phases. The freedom to rephase the fields in eqs. (1.1) and (1.2) allows us to take v, v_L, v_R as well as m_ν real and positive. Next, we define $y^2 \equiv |y|^2 e^{i\chi}$, so that $\arg d = \chi$ and the phase of f is then determined by eq. (3.1).

Type I and II contributions to m_ν are defined as

$$m_\nu^I \equiv -\frac{v^2 y^2}{v_R f}, \qquad m_\nu^{II} \equiv v_L f = m_\nu - m_\nu^I. \tag{3.3}$$

Their relative size is determined by the value of d:

$$r_\pm^{II/I} \equiv \frac{m_{\nu\pm}^{II}}{m_{\nu\pm}^I} = -\frac{[1 \pm (1+d)^{1/2}]^2}{d}. \tag{3.4}$$

We identify three possible physical regimes:

(i)	$|d| \ll 1$	$|m_{\nu+}^I| \ll |m_{\nu+}^{II}|$ and $|m_{\nu-}^{II}| \ll |m_{\nu-}^I|$	single type dominance
(ii)	$|d| \sim 1$	$|m_\nu^I| \sim |m_\nu^{II}|$	hybrid seesaw
(iii)	$|d| \gg 1$	$m_\nu^I \approx -m_\nu^{II}$	cancellation regime

Let us discuss these cases in turn.

(i) The dominant seesaw type is I (II) in the case of the f_- (f_+) solution. For $|d| \ll 1$, eq. (3.1) becomes

$$f_- \approx -\frac{v^2 y^2}{v_R m_\nu}, \qquad f_+ \approx \frac{m_\nu}{v_L} + \frac{v^2 y^2}{v_R m_\nu}. \tag{3.5}$$

Therefore a value $|d| \ll 1$ implies that one type of seesaw is dominant, but it does not determine which one. Thus, *both type I and type II dominance limits correspond*
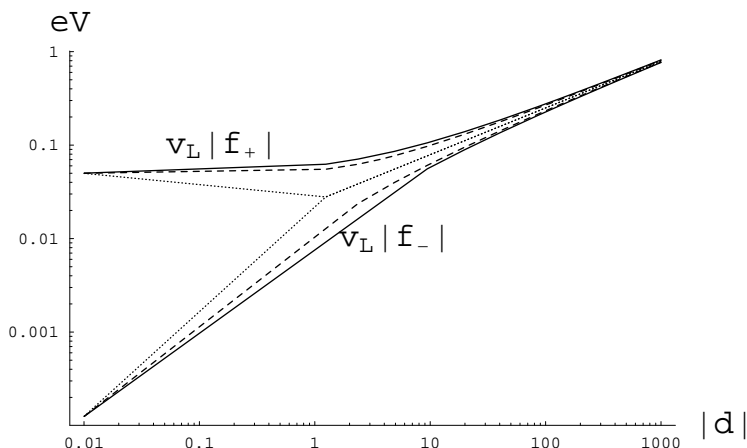
Figure 1: Moduli of the solutions $v_L f_\pm$ of the seesaw formula, as functions of $|d|$. We chose $m_\nu = \sqrt{\Delta m_{\text{atm}}^2} = 0.05\,\text{eV}$. The solid, dashed and dotted curves correspond to $\arg d = 0$, $\pi/2$ and π, respectively. Recall that $m_{\nu\pm}^{II} = v_L f_\pm$ and $m_{\nu\pm}^I = m_\nu - m_{\nu\pm}^{II} = v_L f_\mp$.

to the same condition $|d| \ll 1$. This may look counter-intuitive, as these two limits correspond to apparently opposite conditions $|m_\nu^I| \gg |m_\nu^{II}|$ and $|m_\nu^I| \ll |m_\nu^{II}|$. However, it is easy to see that, when expressed in terms of the "input" parameters only, i.e. when the corresponding solutions for f are substituted, both conditions reduce to $|d| \ll 1$.

A solution f is meaningful (perturbative) only if $|f| \lesssim 1$: for $v_L \ll m_\nu$, the solution f_+ violates perturbative unitarity and should be discarded, so that only f_- is viable (dominant type I). A necessary condition for both solutions to be perturbative is $v_R \gtrsim v^2 |y|^2/m_\nu$, so that either type I or type II contribution to m_ν can dominate. Notice that for $|d| \ll 1$ the expansions (2.4) and (2.6) apply for f_- and f_+, respectively; eq. (3.5) just gives a simplified version of such expansions.

(ii) For $|d| \sim 1$ both seesaw types give sizable contributions to m_ν. Notice that when $d = -1$ (i.e., $4v_L v^2 |y|^2 = v_R m_\nu^2$ and $\chi \equiv \arg y^2 = \pi$), the equality $m_\nu^I = m_\nu^{II} = m_\nu/2$ is realized. This degeneracy point corresponds to the absence of CP violation.[6]

(iii) The case $|d| \gg 1$ corresponds to a cancellation between the two seesaw contributions to m_ν. To bar a too strong ("unnatural") cancellation, one may demand $|m_\nu^{I,II}| \lesssim 1\,\text{eV}$. Eq. (3.3) then implies a lower bound on the scale v_R of $SU(2)_R$ symmetry breaking: $v_R \gtrsim v^2 |y|^2/(|f| \cdot 1\,\text{eV}) \approx |y^2/f| 3 \cdot 10^{13}\,\text{GeV}$. If one makes the additional ("natural") assumption $x \equiv v_R v_L/v^2 \sim 1$, eq. (3.3) will also imply $v_R \gtrsim v^2 |f|/\text{eV} \approx |f| 3 \cdot 10^{13}\,\text{GeV}$.

For illustration, let us assume that the light neutrino mass is given by the atmospheric scale, $m_\nu = 0.05\,\text{eV}$. The moduli $v_L |f_\pm|$ of the solutions of eq. (3.1) as functions of $|d|$ are

---

[6]Note that in general CP violation is present in the one-generation case due to the presence of the Higgs triplets. Although it does not manifest itself at low energies, it can lead to a successful leptogenesis (more on this in section 6.2).
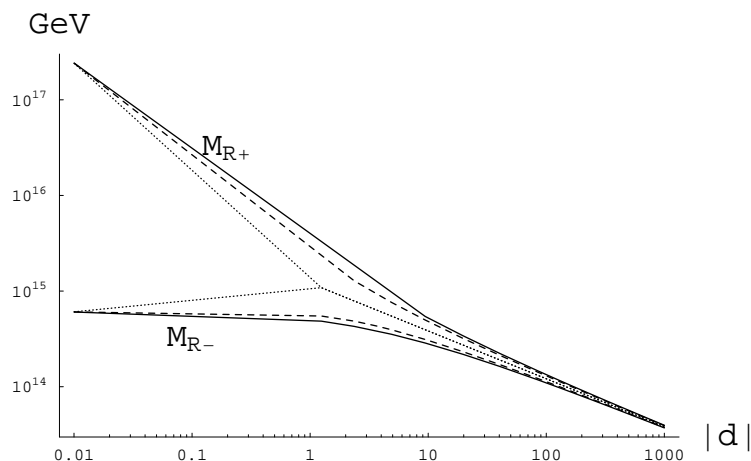
Figure 2: The mass $M_{R\pm} = v_R|f_\pm|$ of the right-handed neutrino for the two solutions $f_\pm$ of the seesaw formula. We chose $m_\nu = \sqrt{\Delta m_{\mathrm{atm}}^2} = 0.05\,\mathrm{eV}$ and $|y| = 1$. For fixed $|d|$, M_R scales as $|y|^2$. The solid, dashed and dotted curves correspond to $\arg d = 0$, $\pi/2$ and π, respectively.

then shown in figure 1. The three regimes (i), (ii) and (iii) correspond to the left, central and right regions of the plot, respectively. The value of $|f_+|$ ($|f_-|$) is a slightly decreasing (increasing) function of $\arg d = \chi$ between 0 and π, as shown in the figure. For $\chi = \pi$ and $|d| \geq 1$ the moduli of the two solutions coincide.

The mass of the RH neutrino is given by

$$M_{R\pm} \equiv v_R|f_\pm| = \frac{v^2}{m_\nu}\left|\frac{2y^2[1 \pm (1 + d)^{1/2}]}{d}\right|. \tag{3.6}$$

In figure 2 we plot M_R as a function of $|d|$ for $m_\nu = 0.05\,\mathrm{eV}$ and $|y| = 1$. The mass of the scalar triplet M_Δ is not directly related to the parameters in the seesaw formula. One expects $M_\Delta \sim v^2/v_L \sim v_R$, but very different values are also naturally possible, depending on the details of the mechanism which induces v_L.

Consider now the case of n lepton generations with no mixing (which means that f and y are diagonal in the same basis). This case can be described as n replicas of the one-generation case, each with two solutions $f_\pm^{(i)}$ ($i = 1, \ldots, n$). For each generation, the relative size of type I and II contribution to the light neutrino mass $m_{\nu i}$ depends on the dominance parameter

$$d_i \equiv \frac{v_L}{v_R}\frac{4v^2 y_i^2}{m_{\nu i}^2}, \quad i = 1, \ldots, n. \tag{3.7}$$

For a given value of v_L/v_R, one can have $|d_i| \ll 1$ (single type dominance) for an i-th generation, while at the same time the hybrid seesaw or cancellation regimes may be realized for $j \neq i$. We will see that in the presence of flavor mixing it is much less trivial to assess the relative size of the two seesaw types.

3.2 The case of two lepton generations

In this case in the flavor basis, where the mass matrix of charged leptons is diagonal (say,

$m_l = diag(m_\mu, m_\tau))$, the seesaw formula (1.3) can be explicitly written as

$$\begin{pmatrix} m_{\mu\mu} & m_{\mu\tau} \\ m_{\mu\tau} & m_{\tau\tau} \end{pmatrix} = \begin{pmatrix} f_{22} & f_{23} \\ f_{23} & f_{33} \end{pmatrix} - \frac{1}{xF} \begin{pmatrix} y_{\mu2} & y_{\mu3} \\ y_{\tau2} & y_{\tau3} \end{pmatrix} \begin{pmatrix} f_{33} & -f_{23} \\ -f_{23} & f_{22} \end{pmatrix} \begin{pmatrix} y_{\mu2} & y_{\tau2} \\ y_{\mu3} & y_{\tau3} \end{pmatrix} , \qquad (3.8)$$

where $F \equiv \det f = f_{22}f_{33} - f_{23}^2$. The matrix equation (3.8) is equivalent to the system of three coupled non-linear equations for f_{22}, f_{33} and f_{23}:

$$\begin{aligned}
xF(f_{22} - m_{\mu\mu}) &= f_{33}y_{\mu2}^2 - 2f_{23}y_{\mu2}y_{\mu3} + f_{22}y_{\mu3}^2 , \\
xF(f_{23} - m_{\mu\tau}) &= f_{33}y_{\mu2}y_{\tau2} - f_{23}(y_{\mu3}y_{\tau2} + y_{\mu2}y_{\tau3}) + f_{22}y_{\mu3}y_{\tau3} , \\
xF(f_{33} - m_{\tau\tau}) &= f_{33}y_{\tau2}^2 - 2f_{23}y_{\tau2}y_{\tau3} + f_{22}y_{\tau3}^2 .
\end{aligned} \qquad (3.9)$$

To solve the system (3.9), we use the following procedure. Let us define $f' = f/\sqrt{\lambda}$, $m' = m/\sqrt{\lambda}$ and $y' = y/\sqrt{\lambda}$, where λ is an as yet arbitrary complex number. The scaling law was chosen in such a way that in terms of the primed variables the system of equations for f'_{ij} has the same form as eq. (3.9). Next, we fix the value of λ by requiring $F' \equiv \det f' = 1$. The system of equations for f'_{ij} then becomes linear and can be readily solved. Expressing the primed variables back through the unprimed ones and substituting them into the condition $F'(\lambda) = 1$, one obtains a 4th order polynomial equation for λ. In general, it has four complex solutions λ_i ($i = 1, \ldots, 4$), leading to four allowed matrix structures f_i. Notice that $F_i \equiv \det f_i = \lambda_i$. This procedure proves that eq. (3.8) has four solutions for a generic structure of y and, therefore, it generalizes (for the case $n = 2$) the proof given in section 2.3 for symmetric y.

We now present the general analytic solution in the case where the matrix y is invertible and symmetric, so that the duality holds. As follows from eq. (2.8), when $y = y^T$ we can choose the basis where y is diagonal: $y_{\mu3} = y_{\tau2} = 0$, $y_{\mu2} \equiv y_2$, $y_{\tau3} \equiv y_3$. With a little abuse of notation, we will still denote the matrix elements of m and f in the new basis as $m_{\alpha\beta}$ and f_{ij}. The system (3.9) can be linearized as described above and is easily solved:

$$f = \frac{x\lambda}{(x\lambda)^2 - y_2^2 y_3^2} \begin{pmatrix} x\lambda m_{\mu\mu} + y_2^2 m_{\tau\tau} & m_{\mu\tau}(x\lambda - y_2 y_3) \\ \ldots & x\lambda m_{\tau\tau} + y_3^2 m_{\mu\mu} \end{pmatrix} , \qquad (3.10)$$

where λ is a solution the following quartic equation:

$$\left[(x\lambda)^2 - y_2^2 y_3^2 \right]^2 - x \left[\det m(x\lambda - y_2 y_3)^2 x\lambda + (m_{\mu\mu}y_3 + m_{\tau\tau}y_2)^2 (x\lambda)^2 \right] = 0 . \qquad (3.11)$$

Taking the determinant of the equality $\hat{f} \equiv m - f = -yf^{-1}y/x$, one obtains

$$x^2 \lambda \hat{\lambda} \equiv x^2 F \cdot \hat{F} = y_2^2 y_3^2 , \qquad (3.12)$$

where $\hat{F} \equiv \det \hat{f}$. With the help of this relation, it is straightforward to check that the four solutions f_i defined by eqs. (3.10) and (3.11) form two dual pairs. Actually, the duality makes it easy to express the four solutions of eq. (3.11) in the closed form as follows:

$$x\lambda_i = \frac{1}{4} \left[x \det m + r_\pm \pm \sqrt{2(x \det m)^2 + 4kx + 2r_\pm x \det m} \right] , \qquad (3.13)$$

where

$$k \equiv m_{\mu\mu}^2 y_3^2 + 2m_{\mu\tau}^2 y_2 y_3 + m_{\tau\tau}^2 y_2^2, \qquad r_\pm = \pm\sqrt{(x \det m)^2 + 4kx + 16y_2^2 y_3^2}. \qquad (3.14)$$

One pair of dual solutions corresponds to r_+ and the other one to r_-; within each pair, a solution is distinguished from its dual by the sign in front of the radical in eq. (3.13). Summarizing, eqs. (3.10) and (3.13) define explicitly the four solutions f_i as functions of the input parameters $m \equiv m_\nu/v_L$, $x \equiv v_L v_R/v^2$ and y.

We can now study the structure of f_i in any regions of parameters of physical interest. To start with, let us consider the limit $y_2 \to 0$. This can be justified if neutrino Yukawa couplings are related to those of charged fermions, so that $y_2 \ll y_3 \lesssim 1$. Then eq. (3.13) can be expanded as

$$x\lambda_{1,3} = \frac{x \det m + r_\pm^0}{2} + \frac{2xy_3 m_{\mu\tau}^2}{r_\pm^0} y_2 + \mathcal{O}(y_2^2), \qquad x\lambda_{2,4} \equiv x\hat\lambda_{1,3} = \frac{y_2^2 y_3^2}{x\lambda_{1,3}}, \qquad (3.15)$$

where $r_\pm^0 = \pm\sqrt{(x \det m)^2 + 4xy_3^2 m_{\mu\mu}^2}$. The solutions for f take the form

$$f_{1,3} = m - f_{2,4}, \qquad f_{2,4} = \begin{pmatrix} 0 & \dfrac{y_2 y_3 m_{\mu\tau}}{x\lambda_{1,3}} \\ \cdots & -\dfrac{y_3^2 m_{\mu\mu}}{x\lambda_{1,3}} \end{pmatrix} + \mathcal{O}(y_2^2). \qquad (3.16)$$

Since $\lambda_{2,4} = \det f_{2,4}$ are proportional to y_2^2, in the case of the solutions $f_{2,4}$ one RH neutrino mass becomes much smaller than v_R for very small y_2. Notice that, even though $\lambda_{2,4}$ go to zero for $y_2 \to 0$, the matrices $f_{2,4}$ are finite and invertible (and therefore acceptable solutions) for any $y_2 \neq 0$.

Let us define, in analogy with eq. (3.2), a dominance parameter that controls the relative size of type I and type II seesaw contributions:

$$d \equiv \frac{v_L}{v_R} \frac{4v^2 y_3^2}{(m_\nu)_{\tau\tau}^2} = \frac{4y_3^2}{xm_{\tau\tau}^2}. \qquad (3.17)$$

Obviously, other analogous parameters can be defined by replacing $m_{\tau\tau}$ and/or y_3 with the other entries of m and y. A detailed discussion of this issue is postponed to appendix B; here we just study the main features of the dependence of the solutions on d. Figure 3 shows the dependence of the values of $|x\lambda_i|$ on $|d|$ for a specific realistic set of the input parameters. In the region $|d| \ll 1$, there is a solution $|x\lambda_1| \gg |y_i y_j|$, which leads to $f_1 \simeq m$ (dominant type II seesaw), while for the dual solution, one has $|x\lambda_2| \ll |y_i y_j|$ (dominant type I seesaw). In general, the other pair of dual solutions corresponds to hybrid seesaw. Only if the value of $\det m$ is strongly suppressed, one finds $f_3 \simeq f_1$ (dominant type II seesaw) and $f_4 \simeq f_2$ (dominant type I seesaw). In the region $|d| \sim 1$, all four solutions are of hybrid type. Finally, for $|d| \gg 1$ a cancellation between type I and II seesaw contributions to m_ν occurs.

In figure 4 we plot the values $M_{2,3}$ of the two RH neutrino masses (that is, the eigenvalues of $M_R = v_R f$) versus $|d|$ for the same set of the input parameters as in figure 3. In
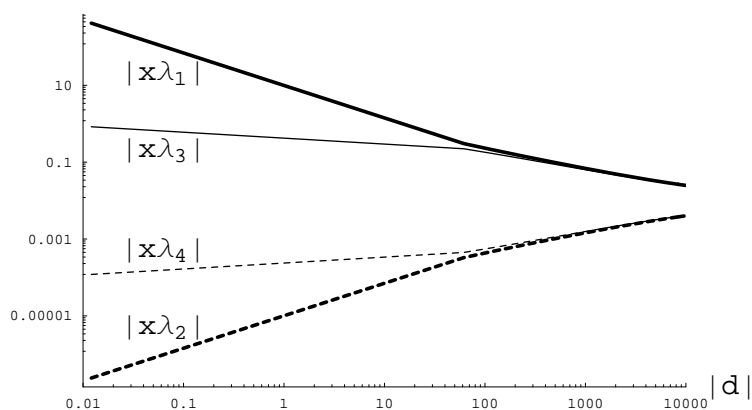
Figure 3: The four solutions $x\lambda_i$ of eq. (3.13) as functions of $|d| \equiv 4y_3^2/(xm_{\tau\tau}^2)$. We chose $m_{\mu\mu} = m_{\tau\tau} = (0.02\,\text{eV})/v_L$ and $m_{\mu\tau} = (0.03\,\text{eV})/v_L$, so that $\theta_{23} = \pi/4$ and $\Delta m_{23}^2 = 2.4 \cdot 10^{-3}\,\text{eV}$. We also took $y_3 = 1$ and $y_2 = 10^{-2}$.
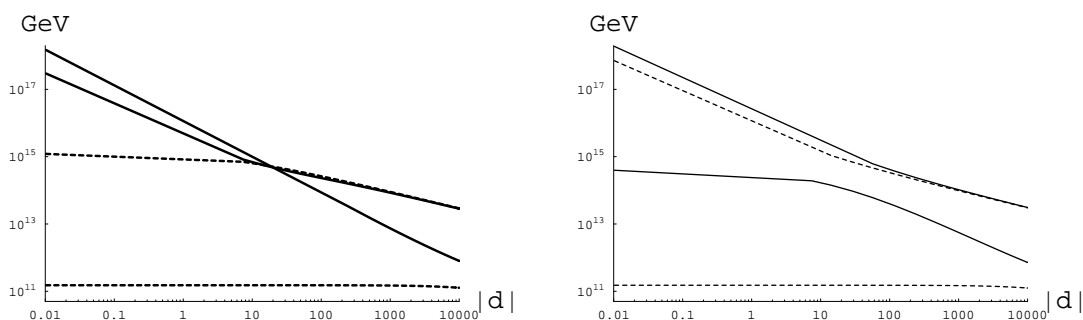

Figure 4: The masses $M_{2,3}$ of the RH neutrinos as functions of $|d|$ for the same set of input parameters as in figure 3. Thick solid (dashed) curves in the left panel correspond to the solution f_1 (f_2), thin solid (dashed) curves in the right panel, to the solution f_3 (f_4).

order to assess the size of $M_{2,3}$, it is useful to define a "seesaw scale" $M_s \equiv v^2/m_3$, which for $m_3 = 0.05$ eV is equal to $6 \cdot 10^{14}$ GeV. Since for $|d| \ll 1$ the solution f_1 (f_2) corresponds to type II (I) seesaw dominance, one has $M_{2,3} \gg M_s$ for f_1 and $M_{2,3} \sim y_{2,3}^2 M_s$ for f_2, as shown in the left panel of figure 4. For the solutions $f_{3,4}$, the situation is intermediate and one finds $M_3 \gg M_s$, $M_2 \sim y_{2,3}^2 M_s$, as shown in the right panel of figure 4. For $|d| \gg 1$, the asymptotic values of M_2 are the same for all four solutions; the same is also true for M_3, with $M_2/M_3 \sim y_2/y_3$. For the solution f_1, one observes a level-crossing of the two RH neutrino masses for $|d| \sim 20$. One could further characterize the sector of RH neutrinos by studying the dependence of their mixing on d.

3.3 The case of three lepton generations

In this case, the symmetric matrix f contains 6 independent elements f_{ij}. The LR symmetric seesaw relation (1.3) can be written as a system of 6 nonlinear coupled equations for

these elements, analogous to that in eq. (3.9). Here we solve this system analytically for the case of a symmetric neutrino Yukawa coupling matrix y, for which the duality holds. Using the freedom to work in the basis where y is diagonal, $y = diag(y_1, y_2, y_3)$, one can write this system as

$$xF(f_{ij} - m_{ij}) = y_i y_j F_{ij}, \qquad (3.18)$$

where $F \equiv \det f$ and

$$F_{ij} \equiv \frac{1}{2} \epsilon_{ikl} \epsilon_{jmn} f_{km} f_{ln}. \qquad (3.19)$$

The six corresponding equations for the dual matrix $\hat{f} \equiv m - f$ are

$$x\hat{F}(\hat{f}_{ij} - m_{ij}) = -x\hat{F} f_{ij} = y_i y_j \hat{F}_{ij}, \qquad (3.20)$$

where $\hat{F} \equiv \det \hat{f}$ and

$$\begin{aligned}
\hat{F}_{ij} &\equiv \frac{1}{2} \epsilon_{ikl} \epsilon_{jmn} \hat{f}_{km} \hat{f}_{ln} = F_{ij} - T_{ij} + M_{ij}, \\
T_{ij} &\equiv \epsilon_{ikl} \epsilon_{jmn} f_{km} m_{ln}, \quad M_{ij} \equiv \frac{1}{2} \epsilon_{ikl} \epsilon_{jmn} m_{km} m_{ln}.
\end{aligned} \qquad (3.21)$$

Taking the determinant of the equality $\hat{f} = -yf^{-1}y/x$, one proves immediately that

$$x^3 F\hat{F} = -y_1^2 y_2^2 y_3^2, \qquad (3.22)$$

which is an analogue of eq. (3.12).

To solve the system (3.18) for f, we make use of a procedure similar to that employed in the 2-generation case. Let us define $f' = f/\lambda^{1/3}$, $m' = m/\lambda^{1/3}$ and $y' = y/\lambda^{1/3}$ where λ is determined from the condition $F'(\lambda) \equiv \det f'(\lambda) = 1$. At this point the left-hand (right-hand) side of equations (3.18), written in terms of the primed variables, becomes linear (quadratic) in f'_{ij}. Using the duality (eqs. (3.20)–(3.22)), the quadratic terms, F'_{ij}, can be rewritten as

$$F'_{ij} = \frac{(y_1' y_2' y_3')^2}{x^2 y_i' y_j'} f'_{ij} - M'_{ij} + T'_{ij}. \qquad (3.23)$$

This allows one to linearize the system for f'_{ij}:

$$[x^3 - (y_1' y_2' y_3')^2] f'_{ij} - x^3 m'_{ij} = x^2 y_i' y_j' (T'_{ij} - M'_{ij}). \qquad (3.24)$$

After some algebra, the solution for $f = \lambda^{1/3} f'$ can be written in a rather compact form (to be compared with eq. (3.10)):

$$f_{ij} = \frac{\lambda^2 \left[(\lambda^2 - Y^2)^2 - Y^2 \lambda \det m + Y^4 S \right] m_{ij} + \lambda \left(\lambda^4 - Y^4 \right) A_{ij} - Y^2 \lambda^2 (\lambda^2 + Y^2) S_{ij}}{(\lambda^2 - Y^2)^3 - Y^2 \lambda^2 (\lambda^2 - Y^2) S - 2Y^2 \lambda^3 \det m}, \qquad (3.25)$$

where

$$Y^2 \equiv \frac{(y_1 y_2 y_3)^2}{x^3}, \quad S \equiv \sum_{k,l=1}^{3} \left(\frac{m_{kl}^2 x}{y_k y_l} \right), \quad A_{ij} \equiv \frac{y_i y_j M_{ij}}{x}, \quad S_{ij} \equiv \sum_{k,l=1}^{3} \left(m_{ik} m_{jl} \frac{m_{kl} x}{y_k y_l} \right). \qquad (3.26)$$

The value of λ is determined from the equation

$$F(\lambda) \equiv \det f(\lambda) = \lambda\,, \tag{3.27}$$

where $f(\lambda)$ was defined in eq. (3.25). Eq. (3.27) turns out to be an 8th order polynomial equation for λ,[7] which has in general eight complex solutions. Defining

$$A \equiv \sum_{k,l=1}^{3} \left(\frac{y_k y_l M_{kl}^2}{x} \right)\,, \tag{3.28}$$

this equation reads

$$\left[(\lambda^2 - Y^2)^2 - Y^2\lambda^2 S \right]^2 - \lambda^2(\lambda^2 + Y^2)^2 A - Y^2\lambda^4(\det m)^2 \\ -\lambda \left[\lambda^6 + Y^2\lambda^2(\lambda^2 - Y^2)(5 + S) - Y^6 \right] \det m \,=\, 0, \tag{3.29}$$

to be compared with eq. (3.11). The eight solutions of eq. (3.29) form four dual pairs λ_i, $\hat{\lambda}_i$ $(i = 1, \ldots, 4)$. The duality is described by the relation $\lambda_i\hat{\lambda}_i = -Y^2$, which follows from eq. (3.22). As a consequence, eq. (3.29) can be rewritten as

$$0 = \prod_{i=1}^{4} (\lambda - \lambda_i)\left(\lambda + \frac{Y^2}{\lambda_i} \right) = \prod_{i=1}^{4}(\lambda^2 - z_i\lambda - Y^2), \quad z_i \equiv \lambda_i - \frac{Y^2}{\lambda_i}\,. \tag{3.30}$$

By comparing eqs. (3.29) and (3.30), it is easy to verify that z_i are the roots of the following quartic equation:

$$z^4 - \det m\, z^3 - (2Y^2 S + A)z^2 - Y^2(8 + S)\det m\, z + Y^2[Y^2 S^2 - 4A - (\det m)^2] = 0\,. \tag{3.31}$$

Thus, although the general order 8 algebraic equation does not admit analytic solutions, the equation for λ can be solved in radicals since, due to duality, it reduces to a quartic equation in z. For the latter a general solution in radicals is known, though complicated. The 8 solutions for f are obtained by plugging into eq. (3.25) the values of λ given by

$$\lambda_i(\hat{\lambda}_i) = \frac{z_i \pm \sqrt{z_i^2 + 4Y^2}}{2}\,, \quad (i = 1, \ldots, 4)\,, \tag{3.32}$$

where the sign in front of the radical distinguishes between λ_i and $\hat{\lambda}_i$.

Given the complicated algebraic form of the solutions for f, it may seem a hopeless task to quantify the dominance of one or the other seesaw type in the eight cases. However, it turns out that the techniques developed for analyzing this issue in the one- and two-generation cases can be generalized here. The full details of the analysis will be given in appendix B. Here we only describe the generic case, in which the relative size of type I and type II contributions does not change much from one element of the matrix m to another and, in addition, no special cancellations among the entries of m, such as leading to $|\det m| \ll |m_{\alpha\beta}|^3$, occur. Then, in terms of the dominance parameter $|d|$ defined in

---

[7]To recognize that, one has to use duality relations to identify and cancel a common polynomial factor in the numerator and denominator of $F(\lambda)$.

eq. (3.17), the classification goes as follows: for $|d| \ll 1$ there is one pair of dual solution with one seesaw type dominance, while the other three pairs correspond to hybrid seesaw. All three RH neutrino masses are generically larger than v^2/m_i in the solution with type II dominance, only two of them are larger than v^2/m_i in three of the hybrid solutions, only one mass satisfies this condition in the corresponding three duals, and finally all RH neutrino masses are of the order of v^2/m_i in the case of dominant type I seesaw. When $|d| \gtrsim 1$, all 8 solutions for f lead to hybrid seesaw.

In the next two subsections we will specialize our general analytic solution to some physically interesting limits. The reader more interested in numerical examples may proceed directly to section 4.

3.3.1 Limit of hierarchical neutrino Yukawa couplings

A considerable simplification of the general solution for f occurs when type I contributions to the elements of m_ν, proportional to y_1, are negligible, i.e. in the limit $y_1 \ll 1$. In fact, this limit is physically well motivated, in view of the tininess of the Yukawa couplings of the charged fermions of first generation. Strictly speaking, for $y_1 = 0$ the matrix y is not invertible, so that the duality of solutions for f does not hold. Nonetheless, eq. (3.25) is valid for any small but non-zero value of y_1 and therefore can be expanded in powers of y_1. Assuming that λ is finite in the limit $y_1 \to 0$, one finds

$$
f = \begin{pmatrix} m_{ee} & m_{e\mu} & m_{e\tau} \\[2mm] \cdots & \dfrac{m_{\mu\mu} + \dfrac{y_2^2}{x\lambda}\left(M_{22} - \dfrac{y_3^2 m_{ee} m_{e\mu}^2}{x\lambda}\right)}{1 - \dfrac{y_2^2 y_3^2 m_{ee}^2}{(x\lambda)^2}} & \dfrac{m_{\mu\tau} + \dfrac{y_2 y_3 m_{e\mu} m_{e\tau}}{x\lambda}}{1 + \dfrac{y_2 y_3 m_{ee}}{x\lambda}} \\[6mm] & & \dfrac{m_{\tau\tau} + \dfrac{y_3^2}{x\lambda}\left(M_{33} - \dfrac{y_2^2 m_{ee} m_{e\tau}^2}{x\lambda}\right)}{1 - \dfrac{y_2^2 y_3^2 m_{ee}^2}{(x\lambda)^2}} \\[6mm] \cdots & \cdots & \end{pmatrix} + \mathcal{O}(y_1). \tag{3.33}
$$

The e-row of the mass matrix of light neutrinos m directly determines the first row of f (pure type II seesaw), whereas the $2-3$ sector of f depends on the interplay of the elements of the e-row and $\mu\tau$-block of m, as well as on the values of y_2, y_3 and $x\lambda$.

Consider now the behavior of λ in the limit $y_1 \to 0$. For $y_1 = 0$, the 8th order polynomial equation (3.29) reduces to

$$
\lambda^4 \left\{ [(x\lambda)^2 - m_{ee}^2 y_2^2 y_3^2]^2 - x \left[\det m (x\lambda - m_{ee} y_2 y_3)^2 x\lambda + (M_{22} y_2 + M_{33} y_3)^2 (x\lambda)^2 \right] \right\} = 0. \tag{3.34}
$$

The zeros of the term in curly brackets in eq. (3.34) are (to leading order) the four solutions λ which are finite for $y_1 \to 0$. The four duals $\hat{\lambda} = -y_1^2 y_2^2 y_3^2/(x^3 \lambda)$ vanish in this limit, so that for them eq. (3.33) does not hold. However, the corresponding solutions can be obtained by duality: $\hat{f} = m - f$ with f given in eq. (3.33). By expanding directly eq. (3.25), it is easy to check that $\hat{f}_{11} \sim y_1^2$, $\hat{f}_{12,13} \sim y_1$, while the other entries are finite. As a consequence, one RH neutrino mass is much smaller than v_R since it is proportional to y_1^2.

When $y_1 = 0$, one has only four (instead of eight) solutions, since their duals become singular. From the comparison of the term in curly brackets in eq. (3.34) with eq. (3.11), a strong analogy with the pure two-generation case becomes evident. In fact, a *different duality* among the four remaining solutions is present: if $\lambda \neq 0$ satisfies eq. (3.34), also $\tilde{\lambda} \equiv y_2^2 y_3^2 m_{ee}^2 / (x^2 \lambda)$ does, and it corresponds to $\tilde{f} \equiv \tilde{m} - f$, where $\tilde{m}_{\alpha\beta} = m_{\alpha\beta} + m_{e\alpha} m_{e\beta} / m_{ee}$. There are two pairs of such solutions.

A very simple yet non-trivial scenario corresponds to the case when type I contributions to m_ν proportional to y_2 are also negligible ($y_2 \to 0$). In this limit the relation $f_{\alpha\beta} = m_{\alpha\beta}$ (pure type II seesaw) holds for all the entries of f but f_{33} [22, 20]. There are two solutions for f_{33}, defined by

$$(f_{33} - m_{\tau\tau})^2 + \frac{\det m}{M_{33}}(f_{33} - m_{\tau\tau}) - \frac{y_3^2}{x} = 0. \tag{3.35}$$

In particular, this equation admits $|f_{33}| \gg |m_{\tau\tau}| \approx |m_{\mu\mu}|$, that is, f with hierarchical structure can lead to m_ν with large $2 - 3$ mixing.

3.3.2 Limit of hierarchical light neutrino masses

When the mass spectrum of light neutrinos has normal (inverted) ordering, the lightest neutrino mass m_1 (m_3) can be negligibly small. In this case the determinant of $m \equiv m_\nu/v_L$ vanishes. For $\det m = 0$ (more generically, for $|\det m| \ll |Y|$), eq. (3.29) reduces to a pair of quartic equations:

$$(\lambda^2 - Y^2)^2 - [\pm\sqrt{A}\lambda(\lambda^2 + Y^2) + Y^2 S \lambda^2] = 0. \tag{3.36}$$

If λ_i is a solution of the "sign +" equation, then also Y^2/λ_i is, while $-\lambda_i$ and $-Y^2/\lambda_i$ are solutions of the "sign −" equation. Therefore, two pairs of dual solutions for λ are equal in absolute value and opposite in sign to the other two pairs. Moreover, eq. (3.36) has the same form as eq. (3.11), if one makes the identifications

$$Y^2 \leftrightarrow \frac{y_2^2 y_3^2}{x^2}, \qquad \pm\sqrt{A} \leftrightarrow \det m, \qquad S \leftrightarrow \frac{xk}{y_2^2 y_3^2}, \tag{3.37}$$

where the quantities on the right hand sides refer to the two-generation case. Therefore, the quartic equations (3.36) can be explicitly solved and analyzed as in section 3.2.

Since the solar mass squared difference is much smaller than the atmospheric one, in the case of the normal hierarchy one can neglect, in first approximation, both m_1 and m_2. In this limit m_ν becomes a rank-1 matrix, and one can write

$$m = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}. \tag{3.38}$$

Almost maximal atmospheric mixing implies $b \sim c$ as well as $\theta_{13} \sim a/b$. The solar mass scale is zero in the limit of eq. (3.38), so that the 1-2 mixing is undefined. Notice that plugging eq. (3.38) in eq. (3.26), one finds $A_{ij} = 0$ and $S_{ij} = m_{ij}S$. As a consequence, the expression for f given in eq. (3.25) becomes singular in this limit, since $f \propto m$ and thus is

not invertible. One therefore has to resort to a different approach in order to find f. It turns out that in this limit the seesaw equation (3.18) has an infinite number of solutions. To illustrate this phenomenon, consider for simplicity the case $y_1 = 0$. Eqs. (3.33) and (3.34) cannot be used since they would yield $f = m$, an unacceptable result when the matrix m is not invertible. A direct solution of eq. (3.18) gives in this case

$$f = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 + y_2 \cos\alpha & bc + \sqrt{y_2 y_3} \sin\alpha \\ ac & bc + \sqrt{y_2 y_3} \sin\alpha & c^2 - y_3 \cos\alpha \end{pmatrix}, \qquad (3.39)$$

where α is an arbitrary complex number different from zero. (For $\alpha = 0$ a different ambiguity appears:

$$f = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 \pm y_2 & bc \\ ac & bc & c^2 \pm y_3 \end{pmatrix}, \qquad (3.40)$$

where the two "$\pm$" signs are uncorrelated so that, for this value of α, there are four solutions.) Summarizing, when the matrix m is of rank 1 there is an infinite number of solutions for f.

4. Numerical examples

We have shown that, for arbitrary values of $y_{1,2,3}$, $v_{L,R}$ and of the elements of m_ν, one generically finds eight solutions for the matrix f, defined by eqs. (3.25), (3.32) and (3.31). The allowed values of the input parameters were discussed in section 2.1. Here we present several numerical examples which are realistic, in the sense that they reproduce the observed neutrino oscillation parameters and respect all the other experimental constraints. This will enable us to identify the most interesting features of the allowed structures of f.

1. As a first realistic numerical example that we will use as a benchmark point in the space of the input parameters, let us take

$$m \equiv \frac{m_\nu}{v_L} = \begin{pmatrix} 0 & 0.1 & -0.1 \\ 0.1 & 0.55 & 0.45 \\ -0.1 & 0.45 & 0.55 \end{pmatrix}, \qquad (4.1)$$

which corresponds to the tri-bi-maximal mixing ($\tan^2\theta_{23} = 1$, $\tan^2\theta_{12} = 1/2$, $\tan^2\theta_{13} = 0$), no CP violation and $\Delta m_{\rm sol}^2 / \Delta m_{\rm atm}^2 \approx 0.031$, in agreement with all the current data. The eigenvalues of m are $(-0.1,\ 0.2,\ 1)$, so that the spectrum has the normal hierarchy and the lightest neutrino has CP parity opposite to that of the other two. The above choice of m fixes also the value $v_L \approx \sqrt{\Delta m_{\rm atm}^2} \approx 0.05\,{\rm eV}$ (see eq. (2.2)). We also take $x \equiv v_L v_R / v^2 = 1$, which is a natural value if the dimensionless parameters in the scalar potential are all of order one. This determines $v_R \approx 6 \cdot 10^{14}\,{\rm GeV}$. Finally, we take the hierarchy among the eigenvalues of y to be slightly weaker than that for Yukawa couplings of the charged fermions: $y_1 = 10^{-2}$,

$y_2 = 10^{-1}$, $y_3 = 1$. We neglect possible (CKM-like) rotations between the flavor basis and the basis where y is diagonal.

With these choices, the solutions of the LR seesaw formula are

$$f_1 \approx \begin{pmatrix} -0.001 & 0.10 & -0.14 \\ \ldots & 0.56 & 0.49 \\ \ldots & \ldots & 0.88 \end{pmatrix}, \qquad \hat{f}_1 \approx \begin{pmatrix} 0.001 & -0.005 & 0.04 \\ \ldots & -0.006 & -0.04 \\ \ldots & \ldots & -0.33 \end{pmatrix},$$

$$f_2 \approx \begin{pmatrix} -0.01 & 0.11 & -0.04 \\ \ldots & 0.55 & 0.44 \\ \ldots & \ldots & -0.88 \end{pmatrix}, \qquad \hat{f}_2 \approx \begin{pmatrix} 0.006 & -0.008 & -0.06 \\ \ldots & -0.004 & 0.01 \\ \ldots & \ldots & 1.44 \end{pmatrix},$$

$$f_3 \approx \begin{pmatrix} 0.02 & 0.07 & -0.02 \\ \ldots & 0.61 & 0.30 \\ \ldots & \ldots & 1.58 \end{pmatrix}, \qquad \hat{f}_3 \approx \begin{pmatrix} -0.02 & 0.03 & -0.08 \\ \ldots & -0.06 & 0.15 \\ \ldots & \ldots & -1.03 \end{pmatrix},$$

$$f_4 \approx \begin{pmatrix} 0.01 & 0.08 & 0.08 \\ \ldots & 0.60 & 0.25 \\ \ldots & \ldots & -0.19 \end{pmatrix}, \qquad \hat{f}_4 \approx \begin{pmatrix} -0.01 & 0.02 & -0.18 \\ \ldots & -0.05 & 0.20 \\ \ldots & \ldots & 0.74 \end{pmatrix}.$$

(4.2)

The rounding off in the numerical values of f_{ij} is chosen so as to clearly illustrate the matrix structure of the solutions. We have checked that for the high precision solutions the duality relation $f_i + \hat{f}_i = m$ is satisfied very accurately for each dual pair. The structure (4.1) of m is recovered by plugging f_i or $\hat{f}_i$ back into eq. (1.3), but very accurate values of their entries (not shown in eq. (4.2)) should be used. This requirement of high accuracy is a consequence of the strong hierarchy among y_i and exactly vanishing ee-entry of m.

For the solutions f_i (the first column in eq. (4.2)) the dominant $\mu\tau$-block of m is reflected in a dominant 23-block of f. The solutions $\hat{f}_i$ (the second column in eq. (4.2)) exhibit a hierarchical structure, with a dominant 33-entry; the maximal 2-3 mixing angle in m is generated from a small mixing angle in $\hat{f}_i$. All the solutions f_i and $\hat{f}_i$ have one order one eigenvalue, so that for the heaviest RH neutrino mass one finds $M_3 \sim v_R$. The other two eigenvalues can be as small as $\sim 10^{-3}$. Type II (I) seesaw contributions dominate the entries $m_{e\mu}$, $m_{\mu\mu}$ and $m_{\mu\tau}$ for the solutions $f_{1,2}$ ($\hat{f}_{1,2}$). The other entries receive significant contributions from both seesaw types (hybrid seesaw). This is also true for all the entries of the dual pairs of solutions f_3, $\hat{f}_3$ and f_4, $\hat{f}_4$.

2. The largest uncertainty in the structure of m_ν is due to the unknown absolute mass scale of the light neutrinos. Therefore, it is interesting to study the dependence of the structure of f on this scale. Let us consider the same set of input parameters as above, but change the eigenvalues of m_ν as follows: $v_L(-0.1, 0.2, 1) \to (-m_1, \sqrt{m_1^2 + \Delta m_{\text{sol}}^2}, \sqrt{m_1^2 + \Delta m_{\text{atm}}^2})$. The solutions for f will now depend on m_1, which may vary between zero and $\sim 0.23\,\text{eV}$ (due to the cosmological upper bound $\sum_i m_i \lesssim 0.7\,\text{eV}$). The structures of f given in eq. (4.2) correspond to $m_1 = 0.1 v_L \approx 0.005\,\text{eV}$. In figure 5, we plot the masses $M_{1,2,3}$ of the RH neutrinos,
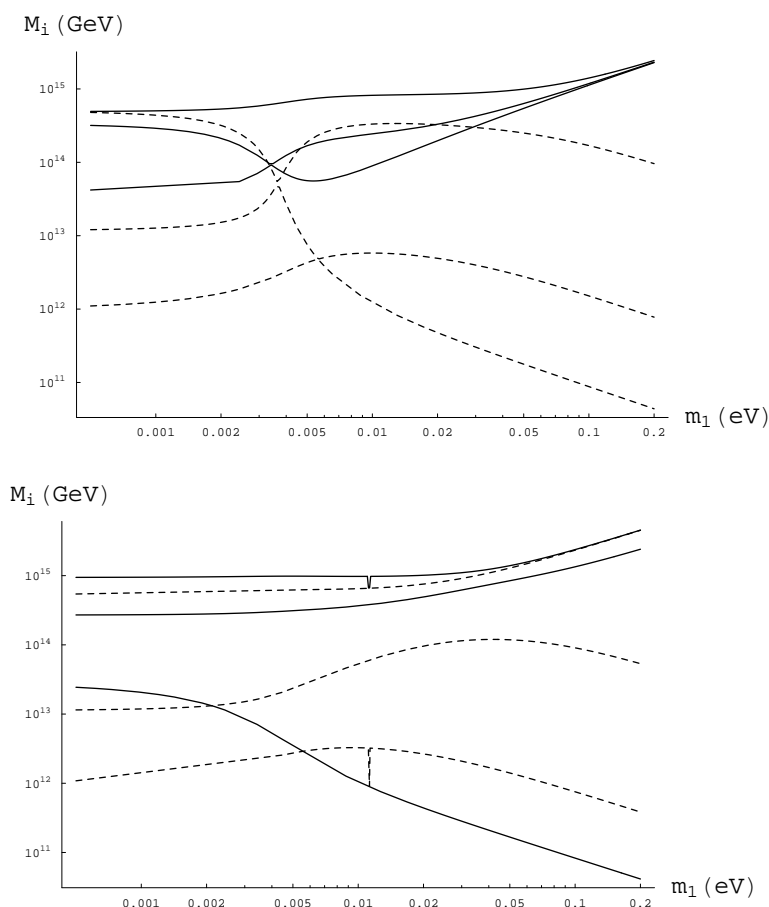
Figure 5: The masses $M_{1,2,3}$ of the three RH neutrinos versus the light neutrino mass scale m_1. In the upper (lower) panel solid curves correspond to the solution f_1 (f_3) and dashed curves to the dual solution $\hat{f}_1$ ($\hat{f}_3$). We chose the tri-bi-maximal mixing, the light neutrino mass spectrum $(-m_1, \sqrt{m_1^2 + \Delta m_{\text{sol}}^2}, \sqrt{m_1^2 + \Delta m_{\text{atm}}^2})$, $v_L \approx 0.051\,\text{eV}$, $v_R \approx 5.9 \cdot 10^{14}\,\text{GeV}$ and $y_{1,2,3} = 10^{-2,-1,0}$. The irregularities around $m_1 = 0.011\,\text{eV}$ in the lower panel are a numerical artifact.

given by the eigenvalues of $M_R \equiv v_R f$, as functions of m_1. We chose for illustration the first and the third pairs of dual solutions.[8]

For the solution f_1, the mass spectrum of RH neutrinos spans at most one order of magnitude, all three masses being rather close to v_R. The two lightest RH neutrinos undergo a level crossing at $m_1 \approx 0.035\,\text{eV}$. Such crossing points were already identified in [23] for the case of pure type I seesaw. In these points the lepton asymmetry generated in the decays of the RH neutrinos can be resonantly enhanced, which may be crucial for reproducing the observed value of the baryon asymmetry of the universe [23]. For $m_1 \gtrsim 0.1\,\text{eV}$ (quasi-degenerate light neutrinos), the three RH neutrino masses also become quasi-degenerate, since f_1 becomes a solution with dominant type

---

[8]Here and below in this section we adopt the numbering of solutions according to which f_i go to the corresponding solutions given in eq. (4.2), when the input parameters approach those chosen for eq. (4.2).

II seesaw, so that $f_1 \approx m$.

The dual solution $\hat{f}_1$ has a substantially different spectrum. The RH neutrino masses span about three orders of magnitude, and two of them can be much smaller than v_R. Level crossings occur between the two heaviest and between the two lightest RH neutrinos. For $m_1 \gtrsim 0.1\,\text{eV}$, by duality $\hat{f}_1$ corresponds to the dominant type I seesaw. The masses M_i of RH neutrinos then scale approximately quadratically with the Dirac-type Yukawa couplings y_i.

For the solution f_3, two RH neutrino masses are close to v_R for all values of m_1, while the lightest RH neutrino mass M_1 decreases by almost three orders of magnitude when m_1 increases in its allowed range. For the dual solution, the RH neutrino masses lie between 10^{15} and 10^{12} GeV. This dual pair corresponds to hybrid seesaw for all values of m_1.

3. The ordering (normal or inverted) is another important information about the light neutrino mass spectrum that is presently missing. Let us consider the input parameters chosen in example 1, but replace the normally ordered spectrum of m with an inverted one: $(-0.1, 0.2, 1) \rightarrow (-1+\epsilon, 1+\epsilon, \epsilon)$, where $\epsilon \equiv (1/4)\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2 \lesssim 0.01$. This amounts to replace eq. (4.1) by

$$m = \frac{1}{6}\begin{pmatrix} -2 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

Then, one pair of dual solutions is given by

$$f_1 \approx \begin{pmatrix} -0.33 & 0.67 & -0.67 \\ \dots & 0.19 & -0.08 \\ \dots & \dots & 1.08 \end{pmatrix}, \qquad \hat{f}_1 \approx \begin{pmatrix} 4\cdot 10^{-5} & -0.001 & 0.001 \\ \dots & -0.01 & -0.08 \\ \dots & \dots & -0.91 \end{pmatrix}, \tag{4.4}$$

and the pair f_2, $\hat{f}_2$ exhibits a qualitatively similar structure. Type II (I) seesaw dominates the entries of the first row of f_1 ($\hat{f}_1$), while hybrid seesaw determines the entries of the 23-block. A third pair of dual solutions is

$$f_3 \approx \begin{pmatrix} -0.35 & 0.51 & -0.52 \\ \dots & -0.77 & 0.68 \\ \dots & \dots & -1.67 \end{pmatrix}, \qquad \hat{f}_3 \approx \begin{pmatrix} 0.02 & 0.15 & -0.15 \\ \dots & 0.95 & -0.84 \\ \dots & \dots & 1.85 \end{pmatrix}, \tag{4.5}$$

and the pair f_4, $\hat{f}_4$ exhibits a qualitatively similar structure. All entries of these matrices receive significant contribution from both type I and type II seesaw.

In figure 6 the RH neutrino spectrum is shown as a function of the absolute mass scale m_3 of light neutrinos. The eigenvalues of m_ν are chosen as $(-\sqrt{m_3^2 + \Delta m_{\text{atm}}^2 - \Delta m_{\text{sol}}^2/2}, \sqrt{m_3^2 + \Delta m_{\text{atm}}^2 + \Delta m_{\text{sol}}^2/2}, m_3)$, which generalizes the choice $v_L(-1+\epsilon, 1+\epsilon, \epsilon)$ that leads to eqs. (4.3)–(4.5). For the solution f_1, the masses of the three RH neutrinos are rather close to each other (within a factor 5) and to
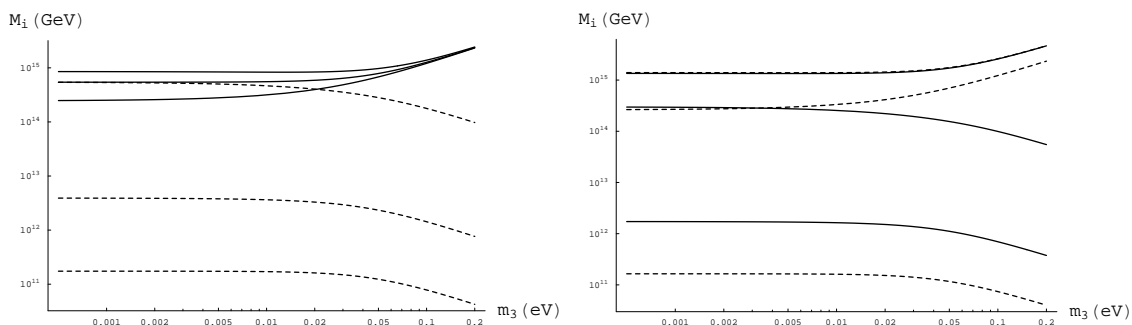
Figure 6: The masses $M_{1,2,3}$ of the three RH neutrinos versus the light neutrino mass scale m_3. We chose the same input parameters as in figure 5, but the inverted light neutrino mass spectrum $(-\sqrt{m_3^2 + \Delta m_{\mathrm{atm}}^2 - \Delta m_{\mathrm{sol}}^2/2}, \ \sqrt{m_3^2 + \Delta m_{\mathrm{atm}}^2 + \Delta m_{\mathrm{sol}}^2/2}, \ m_3)$. In the left (right) panel, the solid curves correspond to the solution f_1 (f_3) and the dashed curves to its dual solution $\hat{f}_1$ ($\hat{f}_3$).

v_R. For the solution $\hat{f}_1$, instead, they are spread over $3 - 4$ orders of magnitude, with the lightest one being around 10^{11} GeV. These features of the mass spectrum of RH neutrinos are rather insensitive to variations of m_3 between zero and 0.23 eV. For both f_3 and $\hat{f}_3$ and in the whole allowed range of m_3, the masses of two RH neutrinos are close to v_R, while the third mass is significantly smaller. Actually, despite the fact that all the entries of f_3 and $\hat{f}_3$ in eq. (4.5) are of order one, the determinants λ_3 and $\hat{\lambda}_3$ turn out to be much smaller than one, corresponding to one RH neutrino mass being much smaller than v_R. Notice also that the value of M_3 is almost the same for these two dual solutions. Contrary to the case of the normal mass ordering, the present set of input parameters does not lead to the level crossing phenomenon.

4. Up to now we were assuming in this section the exact tri-bi-maximal mixing. However, the present uncertainties of the values of the leptonic mixing angles are quite sizable. In particular, only an upper bound on the 1-3 mixing exists, $\sin\theta_{13} \lesssim 0.2$. Measuring this parameter is one of the main goals of the near-future experimental neutrino program. Let us discuss the modifications to the structures of f when a non-vanishing θ_{13} is allowed.

Consider the same input parameters as in example 1, but with $\sin\theta_{13} = 0.1$. Then eq. (4.1) is replaced by

$$m \approx \begin{pmatrix} 0.01 & 0.17 & -0.03 \\ 0.17 & 0.53 & 0.44 \\ -0.03 & 0.44 & 0.56 \end{pmatrix}. \qquad (4.6)$$

The solutions for f are modified slightly with respect to those in eq. (4.2), but their qualitative features remain the same. However, the relative size of type I and II seesaw contributions to a given element of f may change significantly with θ_{13}. The mass spectrum of RH neutrinos in general depends weakly on θ_{13}, as shown in figure 7 for the pairs of dual solutions $f_{1,3}$ and $\hat{f}_{1,3}$: the masses M_i change at most by about
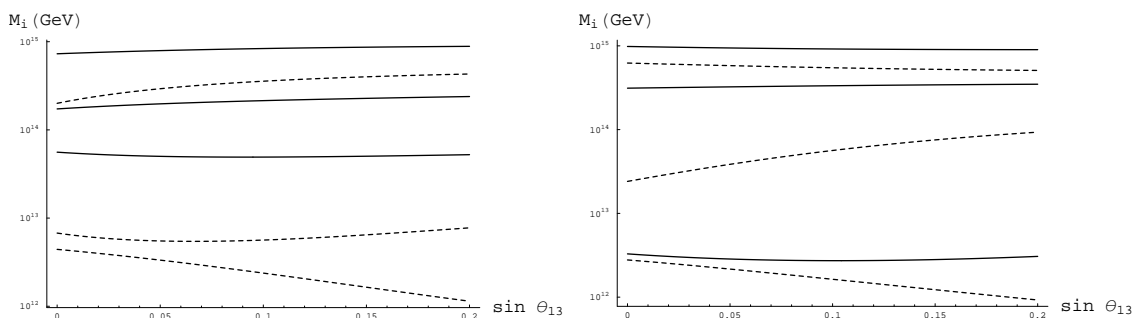
Figure 7: The masses $M_{1,2,3}$ of the three RH neutrinos as functions of $\sin\theta_{13}$. We chose the same input parameters as in figure 5, but fixed $m_1 = 0.005\,\text{eV}$ and allowed non-zero θ_{13}. In the left (right) panel, the solid curves correspond to the solution f_1 (f_3) and the dashed curves to its dual solution $\hat{f}_1$ ($\hat{f}_3$).

a factor of 10 for $\sin\theta_{13}$ varying between 0 and 0.2. This indicates that similar underlying theories at the seesaw scale may result in very different values of θ_{13}.

5. An important experimental and theoretical issue is whether CP is violated in the leptonic sector. The Dirac-type CP-violating phase is associated to the parameter θ_{13} and therefore its effect on the structure of m_ν is small. On the contrary, the two Majorana-type CP-violating phases, that is, the relative complex phases of the eigenvalues $m_{1,2,3}$ of m_ν, can have large effects on the structure of the mass matrix. In particular, the relative phase ρ between m_1 and m_2 affects substantially the parameter $m_{ee} \equiv |(m_\nu)_{ee}|$, which determines the decay rate of nuclei undergoing neutrinoless 2β-decay. Since next generation $2\beta0\nu$ decay experiments will be mostly sensitive to the quasi-degenerate neutrino mass spectrum, we consider here the dependence on ρ assuming three light neutrinos with mass $\sim 0.2\,\text{eV}$.[9]

For definiteness, we once again use the parameters chosen in example 1, except that the mass spectrum of light neutrinos is now taken to be $v_L(e^{-2i\rho},\, 1+\epsilon,\, 1+\eta)$, with $v_L = 0.2\,\text{eV}$, $\epsilon \approx \Delta m^2_{\text{sol}}/(2v_L^2)$ and $\eta \approx \Delta m^2_{\text{atm}}/(2v_L^2)$. The mass matrix of light neutrinos takes the form

$$m = \mathbb{1}_3 + \frac{1-e^{-2i\rho}}{6}\begin{pmatrix} -4 & 2 & -2 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} + \frac{\epsilon}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} + \frac{\eta}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \quad (4.7)$$

In particular, one finds $m_{ee} \approx v_L\sqrt{1-(8/9)\sin^2\rho}$. For $\rho = 0$ ($\rho = \pi/2$) the CP parity of ν_1 is equal (opposite) to that of the other two light neutrinos.

---

[9]In the basis where y is real and diagonal, the matrix m in the seesaw formula depends on 6 CP-violating phases, the three low energy ones plus three phases that are not observable at low energies, but may be relevant for the reconstruction of f. They can be taken into account by allowing y_i to be complex. In the examples considered here we set them equal to zero.
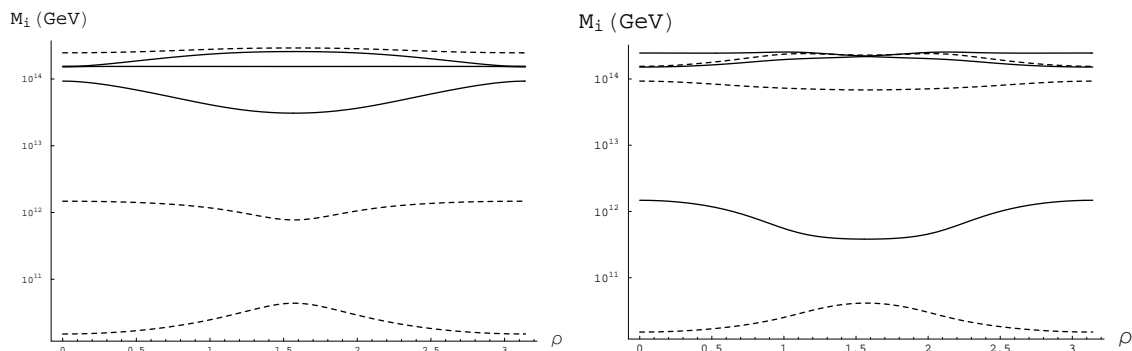
Figure 8: The masses $M_{1,2,3}$ of the three RH neutrinos versus the Majorana-type CP violating phase ρ. In the left (right) panel, the solid curves correspond to the solution f_1 (f_2) and the dashed curves to its dual solution $\hat{f}_1$ ($\hat{f}_2$). We chose the same input parameters as in figure 5, but assumed the quasi-degenerate mass spectrum of light neutrinos $v_L[e^{-2i\rho}, 1 + \Delta m_{\rm sol}^2/(2v_L^2), 1 + \Delta m_{\rm atm}^2/(2v_L^2)]$ with $v_L = 0.2\,{\rm eV}$.

The eight solutions for f depend on the value of the phase ρ in the interval $[0, \pi]$. The dependence of the masses of RH neutrinos on ρ for two pairs of dual solutions is presented in figure 8. The figure is invariant under the transformation $\rho \leftrightarrow \pi - \rho$ because, in the absence of the other complex phases, this transformation is equivalent to taking the complex conjugate of m and therefore of f, so that the masses of RH neutrinos remain unchanged. The dependence of M_i on ρ is generically weak (they change by less than one order of magnitude). The three RH neutrino masses corresponding to the solution f_1 are quasi-degenerate for $\rho = 0$ (type II seesaw dominates: $f_1 \approx m \approx \mathbb{1}_3$), whereas for $\rho = \pi/2$ they are split by an order of magnitude. For the solution $\hat{f}_1$ (dominant type I seesaw) the mass spectrum of RH neutrinos is strongly hierarchical independently of the value of ρ. For the solution f_2, level crossing between M_2 and M_3 occurs at $\rho = \pi/2$.

6. Next, we discuss the dependence of f on the values of the Dirac-type neutrino Yukawa couplings y_i. In the previous examples we assumed the hierarchical values $y_3 = 1$, $y_2 = 0.1$, $y_1 = 0.01$. However, as pointed out in section 2.1, the flavor structure of the Dirac-type Yukawa couplings of neutrinos may be qualitatively different from that for charged fermions.

Let us keep fixed $y_3 \gg y_1$ and vary y_2 in between. In figure 9 we plot M_i as functions of y_2 for the dual pairs $f_{1,3}$ and $\hat{f}_{1,3}$, taking for the other input parameters the same values as in example 1. The left side of the figure corresponds to the limit $y_2 = y_1 = 0.01$, the right side to the limit $y_2 = y_3 = 1$. The central value $y_2 = 0.1$ corresponds to example 1 (more precisely, to the dual pairs $f_{1,3}$ and $\hat{f}_{1,3}$ in eq. (4.2), that is, to the value $m_1 = 0.005\,{\rm eV}$ in figure 5). One can see that for the solutions $f_{1,3}$ the variations of M_i with y_2 are small; this is because type I contribution proportional to y_2 is subdominant. In contrast to this, for the solutions $\hat{f}_{1,3}$ the two lightest RH neutrino masses decrease significantly with decreasing y_2. For the solution $\hat{f}_1$ a level
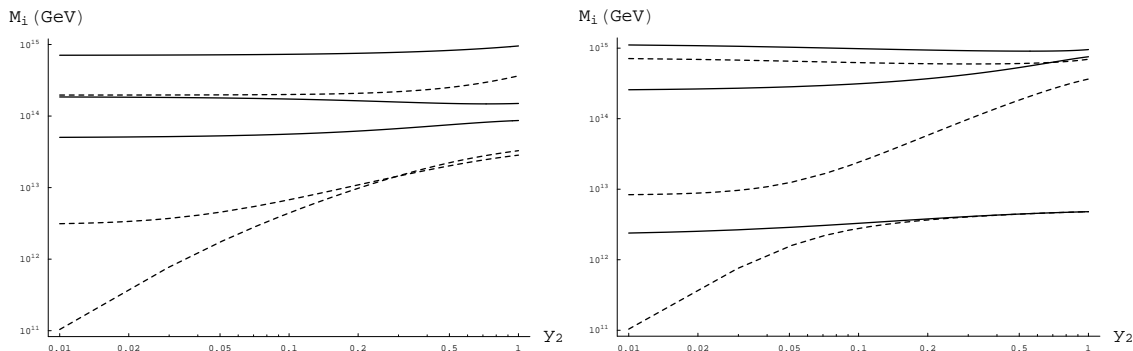
Figure 9: The masses $M_{1,2,3}$ of the three RH neutrinos as functions of y_2. In the left (right) panel the solid curves correspond to the solution f_1 (f_3) and the dashed curves to its dual solution $\hat{f}_1$ ($\hat{f}_3$). We chose the same input parameters as in figure 5, but fixed $m_1 = 0.005\,\mathrm{eV}$ and varied y_2 between $y_1 = 0.01$ and $y_3 = 1$.

crossing occurs at $y_2 \sim 0.3$.

Let us focus on the special case $y_2 = y_3$, which may be motivated by the experimental observation $\nu_3 \approx (\nu_\mu + \nu_\tau)/\sqrt{2}$, that is, $\theta_{13} \approx 0$ and $\theta_{23} \approx \pi/4$. For illustration, we show the structures of f_1 and $\hat{f}_1$:

$$f_1 \approx \begin{pmatrix} -0.0003 & 0.14 & -0.14 \\ \dots & 0.86 & 0.76 \\ \dots & \dots & 0.86 \end{pmatrix}, \qquad \hat{f}_1 \approx \begin{pmatrix} 0.0003 & -0.04 & 0.04 \\ \dots & -0.31 & -0.31 \\ \dots & \dots & -0.31 \end{pmatrix}. \tag{4.8}$$

In this as well as in the other three dual pairs, one finds $f_{12} = -f_{13}$ and $f_{22} = f_{33}$. This is a consequence of the choice $m_{e\mu} = -m_{e\tau}$, $m_{\mu\mu} = m_{\tau\tau}$ and $y_2 = y_3$, as can be directly verified using eq. (3.25). Moreover, this "2-3 symmetry" also connects the dual pairs of solutions to each other. This can be best seen by considering the seesaw formula after the maximal 2-3 rotation:

$$\begin{pmatrix} m_{ee} & \sqrt{2}m_{e\mu} & 0 \\ \dots & m_{\mu\mu} - m_{\mu\tau} & 0 \\ \dots & \dots & m_{\mu\mu} + m_{\mu\tau} \end{pmatrix} = \begin{pmatrix} f_{11} & \sqrt{2}f_{12} & 0 \\ \dots & f_{22} - f_{23} & 0 \\ \dots & \dots & f_{22} + f_{23} \end{pmatrix}$$

$$-\frac{1}{x} \begin{pmatrix} y_1 & 0 & 0 \\ \dots & y_2 & 0 \\ \dots & \dots & y_2 \end{pmatrix} \begin{pmatrix} f_{11} & \sqrt{2}f_{12} & 0 \\ \dots & f_{22} - f_{23} & 0 \\ \dots & \dots & f_{22} + f_{23} \end{pmatrix}^{-1} \begin{pmatrix} y_1 & 0 & 0 \\ \dots & y_2 & 0 \\ \dots & \dots & y_2 \end{pmatrix}. \tag{4.9}$$

The equation for the 33-entry decouples from the rest of the system and is quadratic in $f_{22} + f_{23}$. One its root is the third eigenvalue common to four solutions for f. The other root is the third eigenvalue common to the four duals. The equation for the 1-2 block has the structure of the LR symmetric seesaw in the case of two generations. As a consequence, there are four solutions for f_{11}, f_{12} and $f_{22} - f_{23}$. Each of them can be combined with one or the other solution for $f_{22} + f_{33}$, giving two solutions
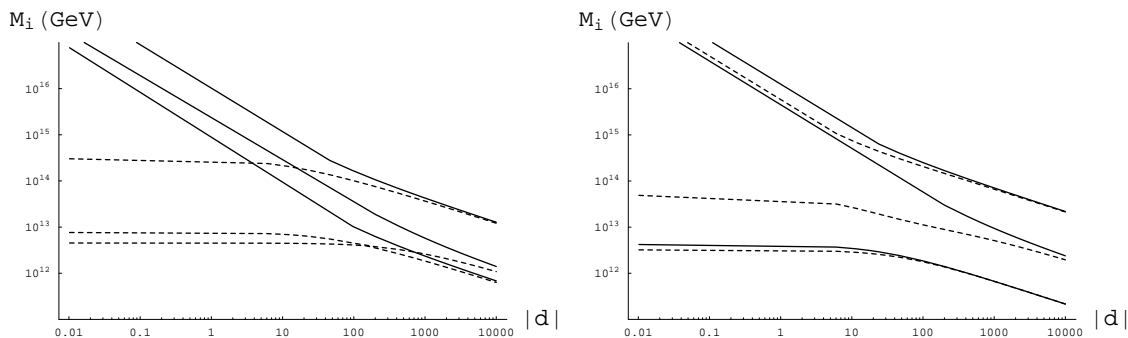
– 25 –

Figure 10: The masses $M_{1,2,3}$ of the three RH neutrinos as functions of the dominance parameter $|d| \equiv |4v^2 y_3^2 / (m_\nu)_{\tau\tau}^2| v_L/v_R$. We chose the same input parameters as in figure 5, but fixed $m_1 = 0.005\,\text{eV}$ and allowed v_R to vary, so that $d = 7.8 \cdot 10^{15}\,\text{GeV}/v_R$. In the left (right) panel, the solid curves correspond to the solution f_1 (f_3) and the dashed curves to its dual $\hat{f}_1$ ($\hat{f}_3$).

which have the same first and second eigenvalues. Note that, despite the presence of this "2-3 symmetry" in the neutrino sector in the considered example, charged leptons break this symmetry badly because $m_\mu \neq m_\tau$.

7. Finally, let us examine the dependence of the masses of the RH neutrinos on the dominance parameter d defined in eq. (3.17). For the one- and two-generation cases we illustrated this dependence in figures 2 and 4, respectively. We now fix all the input parameters as in example 1, but allow x to be different from 1. This gives $d \approx 13/x \approx 8 \cdot 10^{15}\,\text{GeV}/v_R$. Notice that eqs. (3.25) and (3.29) imply that the dependence of f on x amounts to a common rescaling of the three Dirac-type Yukawa couplings: $f(x, y_i) = f(1, y_i/\sqrt{x})$.

In figure 10 we plot the masses M_i of the three RH neutrinos as functions of d for the dual pairs of solutions $f_{1,3}$ and $\hat{f}_{1,3}$. Since M_i are given by the eigenvalues of $v_R f$, they generally decrease as $v_R \propto 1/d$ with increasing d. However, this is not the case when type I seesaw dominates, since in this case $f \propto 1/v_R$ and M_i tend to constant values, as it was already shown in [13]. Such type I seesaw dominance is realized for $d \ll 1$ for the solution $\hat{f}_1$ (left panel in figure 10). Correspondingly, type II seesaw dominates in f_1, so that M_i grow linearly with v_R and the mass spectrum of the RH neutrinos is proportional to that of light neutrinos. As discussed at the end of section 3.3 (see also appendix B), the other three dual pairs exhibit intermediate features for $d \ll 1$, with only M_1 (M_1 and M_2) tending to a constant for the solutions $f_{2,3,4}$ ($\hat{f}_{2,3,4}$), as shown in the right panel of figure 10. Large d corresponds, instead, to a cancellation between type I and II seesaw contributions to m_ν, that is, $f_i \approx -\hat{f}_i$, so that the mass spectra of the RH neutrinos for each pair of dual solutions tend to coincide, as can be seen on the right sides of the plots in figure 10. Further discussion of the dependence of the mass spectrum of RH neutrinos on v_R can be found in [13].
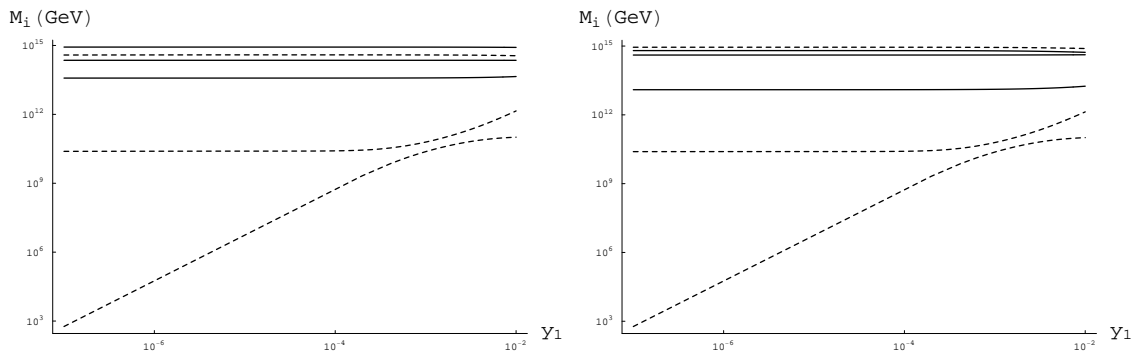
Figure 11: The masses $M_{1,2,3}$ of the three RH neutrinos as functions of y_1 for two dual pairs of solutions. We chose $y_2 = 0.01$, $m_1 = 0.005\,\text{eV}$, $\sin\theta_{13} = 0.1$, while the other input parameters are the same as in figure 5. In both panels dashed curves correspond to the solution with M_1 vanishing in the limit $y_1 \to 0$ and solid curves to the dual solution.

4.1 Right-handed neutrinos at TeV scale

The seesaw mechanism is not accessible to direct experimental tests unless the mass scale of the seesaw particles is as low as $\sim \text{TeV}$. Therefore, from the phenomenological point of view, it is interesting to investigate what regions in the space of the input parameters could lead to new particles with masses at this scale. In this section we present a number of examples where such a scenario is realized.

(1) One vanishing or very small Dirac-type neutrino Yukawa coupling ($y_1 \to 0$). This limit was studied analytically in section 3.3.1, where we showed that in this case four of the solutions for f have one eigenvalue vanishing as y_1^2. We illustrate this phenomenon numerically in figure 11, where y_3 and y_2 have fixed values (1 and 0.01, respectively), whereas y_1 is varied. As expected, in each pair of dual solutions there is one with M_1 decreasing as y_1^2. The other RH neutrino masses are practically independent of y_1 when it becomes smaller than $\sim 10^{-4}$. Notice that for $y_1 \sim 10^{-6}$ the mass of the lightest RH neutrino N_1 is already as small as $\sim 10\,\text{TeV}$. However, its Yukawa couplings to the SM leptons are also suppressed as y_1 and therefore there is no hope to detect N_1 through these couplings. For yet smaller values of y_1, N_1 could be in principle so light as to play a role in cosmology or in active-sterile neutrino oscillations.

(2) Lowering of the LR symmetry breaking scale v_R down to $\sim \text{TeV}$. This is allowed by direct searches of RH currents, as discussed in section 2.1. Such a scenario is not viable for generic ("natural") values of the Yukawa couplings, since it would result in the masses of light neutrinos which are much larger than the observed ones. Nonetheless, in view of the unique phenomenological possibility to directly test the breaking of the LR symmetry and the seesaw mechanism, it is worth accepting the necessary tuning of Yukawa couplings and investigate its consequences.

The simplest phenomenologically viable scenario with $v_R \sim \text{TeV}$ corresponds to choosing tiny values for all the Dirac-type neutrino Yukawa couplings y_i. This allows one to avoid conflict with the low-energy neutrino data while maintaining the perturbative unitarity of the solutions for f, $f_{ij} \lesssim 1$. Notice that the seesaw formula is invariant with respect to a simultaneous decrease of y_i and x provided that y_i^2/x remains unchanged. Consider the same choice of the input parameters as in example 1 of section 4, except that the values $x = 1$, $y_3 = 1$ are replaced with $x = 10^{-10}$, $y_3 = 10^{-5}$, keeping v_L and the hierarchy between y_i unchanged. Then the solutions for f are unchanged, but since $v_R \approx 6 \cdot 10^4 \, \text{GeV}$ in this case, the masses of RH neutrinos are smaller than those in example 1 of section 4 by a factor 10^{10}. As one can see in figure 5, this means that the lightest RH neutrino can have a mass $\sim 100 \, \text{GeV}$. Such light RH neutrinos would not be observable through their Dirac-type Yukawa couplings, which are tiny; however, because of the LR mixing, they may interact strongly with the ordinary matter through the Yukawa couplings f with scalar triplets as well as through their coupling to RH gauge bosons. More generally, the RH gauge bosons as well as the non-standard Higgs particles related to the LR symmetry breaking would provide clear phenomenological signatures if light enough. If level crossing occurs between two TeV-scale RH neutrinos, they could still lead to a successful leptogenesis via the resonant enhancement of the produced lepton asymmetry (see section 6.2).

(3) It should be noted that very small values of x, as in scenario 2 above, would indicate an approximate symmetry of the scalar potential, since for a generic choice of order one couplings one expects $v_L v_R \sim v^2$, that is $x \sim 1$. Very interestingly, it is possible to realize a TeV scale LR symmetry breaking even preserving this "naturalness" relation. Indeed, consider a modification of scenario 2 in which v_L is increased by a large factor W and, at the same time, all y_i^2 are reduced by W. Then the solutions f of the seesaw formula maintain the same matrix structure, but are also reduced by an overall factor W. This means, in particular, that the RH neutrino masses are much smaller than v_R. Using the values of the parameters above and taking $W = 10^8$, one finds $M_1 \sim \text{keV}$. Summarizing, if LR symmetry breaking occurs at scales much lower than the Grand Unification scale and the natural relation $v_L v_R \sim v^2$ is preserved, this would indicate that the Dirac-type and triplet Yukawa couplings y and f are all much smaller than one, with the related consequences for phenomenology.

(4) Finally, let us investigate if TeV scale RH neutrinos may be compatible with at least some order one Dirac-type Yukawa couplings and $v_R \sim \text{TeV}$. This would require a strong fine-tuning in the structure of the Yukawa coupling matrices, in order to cancel large contributions to the masses of light neutrinos. To provide a simple example, it is convenient to start in the basis where the light neutrino mass matrix is diagonal, $m_{\text{diag}} \equiv (m_\nu)_{\text{diag}}/v_L = diag(m_1, m_2, m_3)$. Let us define $r^2 \equiv m_1/m_2$ and choose

$$y = \begin{pmatrix} r^2 s & irs & 0 \\ irs & -s & 0 \\ 0 & 0 & y_3 \end{pmatrix}. \tag{4.10}$$

In this case the seesaw equation for f has only two solutions, the reason being as follows. The matrix y in eq. (4.10) has one zero eigenvalue, which reduces the number of solutions for f from eight to four (section 3.3.1). Furthermore, in the basis where y is diagonal, one has $m_{ee} = 0$, which makes two of the four remaining solutions singular and thus unphysical (see eq. (3.34)). The leftover two solutions are

$$f_{1,2} = diag\left(m_1, \ m_2, \ \frac{m_3}{2}\left[1 \pm \sqrt{1 + \frac{4y_3^2}{xm_3^2}}\right]\right).\qquad(4.11)$$

Notice that the entries of y proportional to s cancel exactly in the seesaw formula and therefore do not contribute to the light neutrino masses. As a consequence, the coupling s can be as large as one. In contrast to this, y_3^2/x has to be smaller than one to guarantee $f_{33} \lesssim 1$. To ensure perturbative unitarity of $(f_{1,2})_{ii}$, v_L should be $\lesssim$ eV; therefore a low LR scale $v_R \sim \mathrm{TeV}$ requires $y_3 \lesssim 10^{-5}$.

To consider the structure of the seesaw mechanism in this scenario in the flavor basis, it is sufficient to rotate on the left and on the right by the leptonic mixing matrix U:

$$m \to U^* m_{\mathrm{diag}} U^\dagger, \quad y \to U^* y U^\dagger, \quad f_{1,2} \to U^* f_{1,2} U^\dagger.\qquad(4.12)$$

In this flavor basis, type I contributions to the elements of the light neutrino mass matrix $m_{\alpha\beta}$ are proportional to $U_{\alpha3}^* U_{\beta3}^* y_3^2/x$.

The masses of RH neutrinos are given by the eigenvalues of $f_{1,2}$ (i.e. the diagonal entries in eq. (4.12)) multiplied by $v_R \sim \mathrm{TeV}$. The two possible solutions for f are distinguished by the mass of the third RH neutrino, while $M_{1,2}$ are uniquely determined, with $M_2^2 - M_1^2 = \Delta m_{\mathrm{sol}}^2 v_R/v_L$. Two of the RH neutrinos ($N_{1,2}$) have order one Yukawa couplings to the light neutrinos and charged leptons.

5. Structure of seesaw in specific left-right symmetric models

In this section, we discuss the basic features of models which incorporate type I+II seesaw mechanism. In particular, we analyze their Yukawa sector and derive the seesaw formula for the light neutrino mass in the form given in eq. (1.3). A crucial issue is the structure of the matrix of Dirac-type Yukawa couplings of neutrinos y, which is an input in the bottom-up approach we adopted to reconstruct f.

5.1 Minimal LR symmetric model

We begin by reviewing the structure of the minimal LR model based on the gauge group $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$. It contains the following (color singlet) Higgs multiplets: $\Phi(2,2,0)$, $\Delta_L(3,1,-2)$ and $\Delta_R(1,3,2)$, where in the brackets we indicate the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$ quantum numbers. The leptons are assigned to $L(2,1,-1) = (\nu_L \ l_L)^T$ and $L^c(1,2,1) = (N_L^c \ l_L^c)^T$, and their Yukawa couplings can be written as

$$-\mathcal{L}_Y = \frac{f_L}{2}L^T i\sigma_2 C\Delta_L L + \frac{f_R}{2}L^{cT} i\sigma_2 C\Delta_R L^c + gL^T C\sigma_2\Phi\sigma_2 L^c + hL^T C\Phi^* L^c + h.c.\,,\quad(5.1)$$

where σ_2 is the isospin Pauli matrix and the following conventions are assumed:

$$\Delta_L = \begin{pmatrix} \Delta_L^+/\sqrt{2} & \Delta_L^{++} \\ \Delta_L^0 & -\Delta_L^+/\sqrt{2} \end{pmatrix}, \quad \Delta_R = \begin{pmatrix} \Delta_R^-/\sqrt{2} & \Delta_R^{--} \\ \Delta_R^0 & -\Delta_R^-/\sqrt{2} \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix}. \quad (5.2)$$

Denoting the VEVs of the neutral components of Φ as $v_{1,2}$, for the Dirac-type mass matrices of the charged leptons and neutrinos one finds

$$m_l \equiv v y_e = v_1 g + v_2^* h, \qquad m_D \equiv v y = v_1^* h + v_2 g. \qquad (5.3)$$

At low energies the Dirac-type neutrino Yukawa coupling can be rewritten in the form of eq. (1.2) by defining $\phi^0 \equiv (v_1 \phi_1^{0*} + v_2^* \phi_2^0)/v$, where $v \equiv \sqrt{|v_1|^2 + |v_2|^2} \approx 174$ GeV is the electroweak symmetry breaking parameter.[10] Notice that if the theory is supersymmetrized, the coupling h should be removed from eq. (5.1) due to the requirement of the analyticity of the superpotential. As a consequence, $y = y_e(v_2/v_1) \equiv y_e \tan\beta$ and the two Higgs doublets, $\phi_u \equiv (\phi_1^+ \ \phi_2^0)^T$ and $\phi_d \equiv (\phi_1^0 \ \phi_2^-)^T$, do not mix.[11]

The Majorana mass matrices of neutrinos are generated by the VEVs $v_{L,R}$ of the neutral components of $\Delta_{L,R}$:

$$m_L = v_L f_L, \qquad M_R = v_R f_R. \qquad (5.4)$$

In the limit where the eigenvalues of of M_R are much larger than those of m_D, the mass matrix of light neutrinos takes the form

$$m_\nu \simeq m_L - m_D M_R^{-1} m_D^T = v_L f_L - \frac{v^2}{v_R} y f_R^{-1} y^T, \qquad (5.5)$$

which is known as type I+II seesaw formula.

At this point, a discrete LR symmetry should be introduced if one wants to guarantee the equality of $SU(2)_L$ and $SU(2)_R$ gauge couplings, that is, the asymptotic LR symmetry of the model. There are essentially two possibilities: the discrete transformation can act as a charge conjugation or as a parity transformation. In the previous sections we were implicitly assuming the former possibility (we postpone the discussion of the latter one to appendix C). Then, the discrete LR symmetry acts on matter and Higgs fields as follows:

$$L \leftrightarrow L^c, \qquad \Phi \leftrightarrow \Phi^T, \qquad \Delta_L \leftrightarrow \Delta_R. \qquad (5.6)$$

This yields

$$f \equiv f_L = f_R, \qquad g = g^T, \qquad h = h^T. \qquad (5.7)$$

In this case the mass matrix of light neutrinos has the form given in eq. (1.3). Moreover, eq. (5.7) implies that y is symmetric, so that the seesaw duality is realized. (If, instead,

---

[10]The field ϕ^0 is thus identified with the neutral component of the SM Higgs boson. The orthogonal combination of the neutral scalar fields, $\eta^0 \equiv (-v_2 \phi_1^{0*} + v_1^* \phi_2^0)/v$, has zero VEV and so does not contribute to the fermion masses. Since it mediates flavour changing neutral currents, it must be heavy and so decouples from the low-energy dynamics.

[11]Similarly to eq. (5.3), for quarks one has $m_d \equiv v y_d = v_1 g_q + v_2^* h_q$ and $m_u \equiv v y_u = v_1^* h_q + v_2 g_q$, where the matrices g_q and h_q are in general independent of g and h, and h_q is absent in the SUSY case.

the model contained a Higgs bidoublet Φ' with the transformation law $\Phi' \leftrightarrow -\Phi'^T$ under the discrete symmetry, the corresponding Dirac-type Yukawa coupling y' would be antisymmetric, leading to the seesaw duality as well).

We have seen that in the minimal SUSY LR model the matrix y is proportional to the Yukawa coupling matrix of charged leptons y_e. Therefore, in the basis where y_e is diagonal, also y is, and the ratios of its eigenvalues are m_e/m_μ and m_μ/m_τ. This scenario is ideal for the reconstruction of f, since the input related to the structure of y is completely determined. However, one should keep in mind that this minimal model does not provide a realistic description of the quark sector, as it leads to $m_u \propto m_d$. As follows from eq. (5.3), in the non-SUSY case, the structure of y can be different from y_e. For a recent realistic model, see [24].

Let us briefly consider an upgrade of the minimal LR model to the Pati-Salam gauge group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4)_c$, with a Higgs field in the $(2, 2, 1)$ representation. In this case quark-lepton unification occurs, which for the Yukawa sector amounts to identifying the quark and lepton coupling matrices: $g_q = g$ and $h_q = h$. This implies, in particular, $y = y_u$ and $y_e = y_d$. In this case the eigenvalues of y are determined by the up-type quark masses, while the mixing is given by the CKM matrix. In non-minimal Pati-Salam models, also Higgs fields transforming as $(2, 2, 15)$ may couple to fermions. Their contribution to the lepton masses differs from those for the quarks by the factor -3 because of the different respective values of $(B - L)$. If both bidoublet types exist and contribute significantly to the fermion masses, the direct connection between y and y_u is lost.

5.2 SO(10) models

A natural grand unified embedding of LR symmetric models is provided by SO(10) models. As is well-known, SO(10) accommodates all the standard model fermions as well as RH neutrinos in a single multiplet $\mathbf{16}_F$. Fermion bilinears transform under SO(10) as

$$\mathbf{16}_F \times \mathbf{16}_F = \mathbf{10}_s + \mathbf{120}_a + \mathbf{126}_s \,, \tag{5.8}$$

where the subscript $s(a)$ indicates symmetry (antisymmetry) in the flavor indexes. As a consequence, there are three possible types of Higgs multiplets which may contribute to fermion masses at the renormalizable level: $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ give symmetric contributions, while $\mathbf{120}_H$ gives an antisymmetric one.

The Higgs triplets $\Delta_{L,R}$ are both contained in $\overline{\mathbf{126}}_H$, so that the Majorana mass matrices of left-handed and RH neutrinos originate from the coupling

$$f \, \mathbf{16}_F \mathbf{16}_F \overline{\mathbf{126}}_H \;\ni\; f(LL\Delta_L + L^c L^c \Delta_R) \,. \tag{5.9}$$

One generator of SO(10), known as D-parity, acts as in eq. (5.6). In other words, the discrete LR symmetry discussed above is promoted to an automatic gauge symmetry and this guarantees the proportionality of the two terms on the r.h.s. of eq. (5.9). The Dirac-type neutrino Yukawa coupling y may receive contributions from the Higgs multiplets of all three types, provided that they acquire VEVs in the appropriate direction. In fact, both $\mathbf{10}_H$ and $\mathbf{120}_H$ contain one bidoublet $\Phi \sim (2, 2, 1)$ under the Pati-Salam group, and both

120_H and $\overline{126}_H$ contain one $(2, 2, 15)$ bidoublet. If only 10_H and $\overline{126}_H$ contribute to y, it is symmetric; if only 120_H does, it is antisymmetric.

Consider first the minimal possibility where only one $\overline{126}_H$ multiplet develops a VEV in the $\nu_L N_L^c$ direction, that is, $y \propto f$. In this case both type I and type II seesaw contributions to m_ν are proportional to f. The latter is uniquely determined by m_ν up to an overall factor. There is no seesaw duality in that case, since y is not independent of f and so it is not invariant under $f \to m - f$.

If two (or more) $\overline{126}_{Hi}$ multiplets are introduced, the proportionality between $m_L \equiv v_{L1} f_1 + v_{L2} f_2$ and $M_R \equiv v_{R1} f_1 + v_{R2} f_2$ is in general lost. The matrix y depends on the same Yukawa couplings $f_{1,2}$: $y = -3(v_{126,1}^u f_1 + v_{126,2}^u f_2)$, where $v_{126,i}^u$ is the VEV of the "up-type" Higgs doublet (i.e. of the doublet with non-zero VEV in the up-type isospin direction) contained in the $(2, 2, 15)$ component of $\overline{126}_{Hi}$. Nonetheless, if $v_{R2} = 0$, an effective seesaw duality is realized, with f_1 and f_2 playing the role of f and y, respectively:

$$\tilde{m} = f_1 - \frac{1}{\tilde{x}} \tilde{y} f_1^{-1} \tilde{y} \,, \tag{5.10}$$

where

$$\tilde{m} \equiv \frac{m_\nu - \left(v_{L2} - 18 v_{126,2}^u v_{126,1}^u / v_{R1}\right) f_2}{v_{L1} - (3 v_{126,1}^u)^2 / v_{R1}} \,, \quad \tilde{x} \equiv \frac{v_{L1} v_{R1} - (3 v_{126,1}^u)^2}{v^2} \,, \quad v \tilde{y} \equiv -3 v_{126,2}^u f_2 \,. \tag{5.11}$$

If all the parameters in eq. (5.10) except f_1 were known, one would be able to solve (5.10) for f_1 using the techniques developed in the previous sections for solving the usual seesaw formula (1.3) for f. For three lepton generation this would yield four dual pairs of solutions.

Consider now the scenario with one 10_H and one $\overline{126}_H$ multiplets. Since 10 is a real representation, 10_H^* transform as 10_H under SO(10), so that in general two new Yukawa couplings should be added to eq. (5.9):

$$g\, 16_F 16_F 10_H + h\, 16_F 16_F 10_H^* \,. \tag{5.12}$$

One then finds

$$m_D \equiv v y = v_{10}^u g + v_{10}^{d*} h - 3 v_{126}^u f \,, \tag{5.13}$$

where $v_{10}^{u,d}$ are the VEVs of the up- and down-type Higgs doublets contained in the $(2, 2, 1)$ component of 10_H. Since y depends on f, one cannot regard it as an input for solving the seesaw formula for f. However, one can consider as an input the matrix $v \tilde{y} \equiv v_{10}^u g + v_{10}^{d*} h$; note that in this class of models the couplings to 10_H usually give the dominant contribution to the mass matrices of the charged fermions.

Then, the seesaw formula (1.3) can be written as

$$\tilde{m} = f - \frac{1}{\tilde{x}} \tilde{y} f^{-1} \tilde{y} \,, \quad \tilde{m} \equiv \frac{m_\nu - (6 v v_{126}^u / v_R) \tilde{y}}{v_L - (3 v_{126}^u)^2 / v_R} \,, \quad \tilde{x} \equiv \frac{v_L v_R - (3 v_{126}^u)^2}{v^2} \,. \tag{5.14}$$

This equation can be solved for f as usual, taking $\tilde{m}$, $\tilde{y}$ and $\tilde{x}$ as input parameters. In some models of this class, the GUT symmetry breaking is such that only the Higgs doublets in 10_H develop VEVs (a mixing between these doublets and those in 126_H is induced only

in the presence of $\mathbf{210_H}$ multiplets). In this case $v^u_{126} = 0$, and one recovers $\tilde{m} = m$, $\tilde{y} = y$ and $\tilde{x} = x$. Moreover, one has $y = y_u$, that is, the matrix y is fully determined if we know the Yukawa couplings of up-type quarks. If also $v^d_{126} = 0$, one finds $y_e = y_d$, which fails to reproduce correctly the masses of the down-type fermions of the second and first generations. The corrections coming from the $\overline{\mathbf{126}}_H$ (and therefore proportional to $v^d_{126}f$) may cure this problem. In this case, however, the form of f is constrained not only by the seesaw formula, but also by the values of the masses of charged fermions.

A comment on the number of the Yukawa couplings to $\mathbf{10_H}$ multiplets is in order. If one wants to study minimal models with only one such Yukawa coupling matrix, there are two options: (i) if only one $\mathbf{10_H}$ is introduced and the model is supersymmetric, the coupling h in eq. (5.12) is automatically forbidden by the analyticity of the superpotential; (ii) one can forbid h also in the non-supersymmetric case by assuming that the multiplet $\mathbf{10_H}$ is real. Then only the coupling g contributes to the fermion masses, leading to $y = y_u \propto y_e = y_d$, which needs substantial corrections in order to match the differences among the mass spectra of charged leptons, up- and down-type quarks. This turns out to be extremely constraining, if one tries to fit all the data by adding just one $\overline{\mathbf{126}}_H$ multiplet (see [6] and references therein). If one is willing to consider models with one more Yukawa coupling matrix, in the SUSY case one needs to introduce a second $\mathbf{10_H}$ multiplet (see e.g. [13]), whereas in the non-SUSY case one complex $\mathbf{10_H}$ is sufficient. In this framework the up sector is decoupled from the down sector, but the relation $y_e = y_d$ persists.

Alternatively, instead of adding $\mathbf{10_H}$ to the $\overline{\mathbf{126}}_H$ multiplet, one can explore the possibility where $\mathbf{120_H}$ is added. The couplings in (5.12) are then replaced by

$$g \; \mathbf{16_F 16_F 120_H} + h \; \mathbf{16_F 16_F 120^*_H} \,, \tag{5.15}$$

where the matrices g and h are antisymmetric. Denoting $v^{u(d)}_{120,1}$ the VEVs of the up (down)-type Higgs doublets contained in the $(2,2,1)$ component of $\mathbf{120_H}$, and similarly $v^{u(d)}_{120,15}$ for the $(2,2,15)$ component of $\mathbf{120_H}$, one finds

$$m_D \equiv vy = (v^u_{120,1} - 3v^u_{120,15})g + (v^{d*}_{120,1} - 3v^{d*}_{120,15})h - 3v^u_{126}f \equiv v\tilde{y} - 3v^u_{126}f \,, \tag{5.16}$$

where $\tilde{y}$ is antisymmetric. Analogously to eq. (5.14), the seesaw relation takes the form

$$\tilde{m} = f + \frac{1}{\tilde{x}}\tilde{y}f^{-1}\tilde{y} \,, \qquad \tilde{m} = \frac{m_\nu}{v_L - (3v^u_{126})^2/v_R} \,, \qquad \tilde{x} = \frac{v_L v_R - (3v^u_{126})^2}{v^2} \,. \tag{5.17}$$

This equation can be solved for f, taking $\tilde{m}$, $\tilde{y}$ and $\tilde{x}$ as input parameters. Using the results of appendix A, which apply to the case of antisymmetric $\tilde{y}$, one finds two solutions for f.

Finally, let us discuss models with both $\mathbf{10_H}$ and $\mathbf{120_H}$ multiplets added to $\overline{\mathbf{126}}_H$ (for an early analysis of fermion masses in this case, see [7]). In the search for the minimal realistic SUSY SO(10) model, it has been recently shown [25] that the mass matrices of the charged fermions can be reproduced through $\mathbf{10_H}$ and $\mathbf{120_H}$ Yukawa couplings only, at least in the first approximation (a three generation fit presents some problems [26]). In the case when the coupling f to $\overline{\mathbf{126}}_H$ is also introduced, satisfactory fits of charged fermion and neutrino parameters have been recently obtained [27]. In this scenario there is

no seesaw duality (at least in the form considered here), since y contains both symmetric and antisymmetric contributions. Nonetheless, the non-linearity of the seesaw formula still leads to multiple solutions for f. To take this into account, one should (i) fix the $\mathbf{10}_H$ and $\mathbf{120}_H$ Yukawa couplings by fitting (approximately) the masses of charged fermions; (ii) use the seesaw formula to derive the different structures of f that reproduce a given set of neutrino data; (iii) test which solutions for f provide the small corrections needed to achieve a satisfactory fit of the masses of charged fermions.

6. Further considerations

In this section we briefly discuss the following issues pertaining to our analysis: stability of our results with respect to the renormalization group evolution effects and baryogenesis via leptogenesis.

6.1 Stability of the seesaw formula

Up to now we were assuming that the seesaw formula (1.3) describes accurately the mass of light neutrinos at low energy scales. However, due to the breaking of the discrete LR symmetry at a scale v_{LR}, the renormalization group (RG) evolution effects below this scale can result in a violation of the conditions in eq. (5.7), which in turn would modify the seesaw formula (1.3). A full study of the RG effects in the type I+II LR symmetric seesaw is beyond the scope of this paper; here we constrain ourselves to a classification of possible effects and estimate of their size.

Above the scale v_{LR} the discrete LR symmetry ensures $f_L = f_R$ and $y = y^T$. Below this scale, the fields related by the LR symmetry acquire in general different masses, so that f_L and f_R may evolve differently and y may get asymmetric corrections. For example, if v_{LR} is larger than v_R (the scale of SU(2)$_R$ breaking), logarithmic in v_R/v_{LR} radiative corrections arise.[12] The size of this effect depends on the details of the mass spectra of RH gauge bosons and Higgs particles responsible for the breaking of SU(2)$_R$, which are highly model-dependent. We will therefore not discuss this case any further and will just assume that the conditions $f_L = f_R$ and $y^T = y$ still hold at the scale $v_R = v_{LR}$.

At this point the fate of the seesaw formula resides with the particle spectrum below v_R. If the masses of all three RH neutrinos as well as of Δ_L are close to v_R, they can be integrated out all together, leading to the low-energy effective neutrino mass matrix m_ν of eq. (1.3). If, on the contrary, one or more RH neutrinos and/or Δ_L have masses that are much smaller than v_R, they will contribute to the RG evolution of f_L, f_R and y down to their mass scale. Generically, these corrections induce a splitting between f_L and f_R and an asymmetry in y. One could envisage two possible approaches to determine the effective mass matrix m_ν at the lightest seesaw scale M_s (the smallest among M_1 and M_{Δ_L}). One option is to evolve $f_{L,R}$ and y from v_R down to M_s and then integrate out the RH neutrinos and Δ_L all together. This has the advantage of determining $f_{L,R}$ and y as they enter in the

---

[12]In SO(10) models, the discrete LR symmetry can be actually broken already at Grand Unification scale, even if SU(2)$_R$ is not [28].

seesaw formula (5.5). Another option is to integrate out N_{Ri} and Δ_L each at its own mass scale, redefining iteratively the effective m_ν. Given the β-functions to some finite order in perturbation theory, this second approach should provide a more accurate result.

For type I seesaw models, a detailed analysis of RG effects, including the running between the different mass scales of RH neutrinos, has been performed in [29]. For type I+II seesaw, such an analysis is not yet available. Here we just estimate the order of magnitude of possible effects. One-loop corrections to the matrix elements of f and y can be schematically written as

$$(\delta f, \delta y) \sim \frac{(f^2, y^2)}{16\pi^2} \log \frac{(M_i, M_{\Delta_L})}{v_R}, \tag{6.1}$$

which, for judicious choices of parameters, should be at or below the percent level. For comparison, the present precision of most of the input parameters in m_ν is at about 10%. Moreover, the loops induced by N_{Ri} exchanges are typically proportional to y^2. The lighter N_{Ri}, the smaller the corresponding Yukawa coupling, since $m_\nu \sim y^2/M_{Ri}$. Therefore, for lighter RH neutrinos, a RG evolution over a wider range may be partly compensated by smaller couplings. Similarly, loops induced by Δ_L exchanges are proportional to f^2 and, since $m_\nu \sim v_L f \sim v^2 f/M_{\Delta_L}$, the lighter the Δ_L, the smaller the expected coupling f and the corresponding radiative correction. As a simplified numerical test, we chose certain structures of f and y at v_R and modified their elements in the seesaw formula by order percent corrections. Next, we implemented the usual bottom-up procedure to reconstruct f and found that, among the 8 dual solutions, one reproduces the original structure of f within percent errors. However, this is not the case when strong hierarchies among the entries of f and y are present *and* small matrix elements receive corrections proportional to the large ones. We expect that only in these special regions of the parameter space instabilities may occur and large departures of the reconstructed structures of f from the true ones may arise.

To complete the discussion of RG effects, one should consider the evolution of m_ν from M_s down to the to electroweak scale, where it can be compared with experimental data. If only the standard model (or MSSM) particles contribute to the RG evolution, the running of the mass squared differences and mixing angles is rather small and negligible for the purposes of this paper, except perhaps in the case of quasi-degenerate light neutrinos of same CP parity. This could affect the reconstruction of f in example 5 of section 4.

6.2 Leptogenesis

As we demonstrated above, in the realistic case of three lepton generations there are eight different matrices f_i which, for a given y, result in exactly the same mass matrix of light neutrinos m_ν. A natural question is then how one can discriminate between these eight possible solutions. The seesaw formula (1.3) cannot tell us more than it already did, and a new independent source of information is necessary. Such a source could be provided by the ability of f_i to reproduce the observed baryon asymmetry of the universe.

It is well known that the seesaw mechanism not only explains nicely the smallness of neutrino mass, but also has a built-in mechanism for generating the baryon asymmetry

of the universe through leptogenesis [4]. In this mechanism, first a lepton asymmetry is produced in out-of-equilibrium CP-violating decays of heavy RH neutrinos and/or Higgs triplets, which is then reprocessed into baryon asymmetry by electroweak sphalerons. The amount of the produced lepton asymmetry depends on (i) mass matrix of RH neutrinos M_R (masses, mixing angles and CP phases); (ii) Majorana-type Yukawa coupling of leptons to Higgs triplets; (iii) Dirac-type Yukawa coupling of leptons to Higgs doublets. In this paper, we consider theories with $f_L = f_R \equiv f$, and so the parameters involved in (i) and (ii) essentially coincide. This renders the computation of the lepton asymmetry more predictive than in generic type I+II seesaw scenarios.

The impact of multiple solutions for f on leptogenesis has been already analyzed in a class of SO(10) models in [13] and very non-trivial results were found. Here we will not undertake a quantitative analysis of leptogenesis, but rather will make some general remarks, which may be of guidance for the development of specific models:

- It is recognized by now [20, 30, 31] that the presence of both RH neutrinos and Higgs triplets may lead to leptogenesis scenarios that are qualitatively different from those in pure type I seesaw models. We would like to stress, as a minimal possibility, that leptogenesis can work in models with Δ_L and just one RH neutrino species N_R (this effective pattern occurs, e.g., when the other two RH neutrinos are super-heavy and/or very weakly coupled). Since N_R couples to a unique linear combination L of the flavor eigenstates of lepton doublets, this scenario may be called "flavorless leptogenesis". This is a viable possibility since the relevant interactions of N_R and Δ_L contain an unremovable CP-violating phase (which manifests itself in the one-generation seesaw formula, see section 3.1). A recent study of this scenario can be found in [32].

- In many realistic cases an extra freedom gained by the interplay of type I and type II seesaw terms may turn out to be insufficient to cure the shortcomings of thermal leptogenesis in the pure type I scenario. In particular, it remains true [30, 33] that the decaying particle should generically be very heavy (above $\gtrsim 10^8$ GeV) to produce a sufficient asymmetry, thus requiring large reheating temperature with the associated problems. Also, the hierarchical structure of Yukawa couplings, which is natural in unified scenarios, may strongly suppress the asymmetry. The existence of various solutions for f, in particular of those with a non-hierarchical structure, may alleviate this problem. Further details can be found in [13].

- Given the rich structure of the Higgs sector at the seesaw scale in LR symmetric and SO(10) models, sources of lepton asymmetry other than the decays of RH neutrinos and/or Δ_L into left-handed leptons should not be overlooked. As an example [34], a successful leptogenesis may be due to decays of RH neutrinos into a RH charged lepton and an $SU(2)_L$ singlet charged scalar (contained, e.g., in the $\mathbf{120}_H$ multiplet of SO(10)). The coupling f_R between the RH leptons and Δ_R (see eq. (5.1)) may also affect the evolution of the asymmetry.

- In minimal models with "natural" values of the seesaw scale and of Yukawa couplings, in most regions of parameter space the produced lepton asymmetry tends to be too small. A way out may be provided by the resonant enhancement of the lepton asymmetry which occurs when two RH neutrinos are quasi-degenerate in mass [35]. As we demonstrated in several examples presented in section 4, the crossing of the RH neutrino mass levels indeed occurs in the LR symmetric seesaw for certain choices of input parameters and for some of the solutions f_i. The requirement of level crossing may be a powerful constraint for model building.

7. Discussion and summary

In models with more than one source of neutrino mass, to disentangle different contributions using only low-energy data is not in general possible. The situation is different when these contributions have a common origin, as it is the case in left-right symmetric seesaw models. In a wide class of such models and their partially unified or Grand Unified extensions, the mass matrix of light neutrinos m_ν contains type I and type II seesaw contributions which depend on the same Majorana-like triplet Yukawa coupling matrix f. In this paper we undertook for the first time a thorough phenomenological bottom-up analysis of such a scenario. We have shown that the interplay of type I and type II seesaw terms in m_ν may change our interpretation of neutrino data and provide an insight into the underlying theory at the seesaw scale.

We have adopted an approach in which the mass matrix of light neutrinos m_ν and the matrix of Dirac-type Yukawa couplings y are considered known, and the seesaw relation is solved for the matrix of the Majorana-like Yukawa couplings f, which coincides (up to a constant factor) with the mass matrix of RH neutrinos and deeply characterizes the structure of the underlying theory. To this end, we have developed a linearization procedure which allowed us, for symmetric or antisymmetric y, to solve the seesaw non-linear matrix equation and obtain exact analytic expressions for the matrix f in a compact form.

For symmetric y, the overall number of solutions was shown to be 2^n for n lepton generations. Thus, in the realistic case of three generations, there are eight different matrices f which, for a given y, result in exactly the same mass matrix of light neutrinos m_ν.

We have studied implications of an intriguing duality property [12] of the LR symmetric seesaw mechanism with symmetric or antisymmetric y, which relates pairwise different solutions for the matrix f. The eight solutions of the seesaw equation for f form four dual pairs. We have demonstrated that, if one of the solutions f of the seesaw equation corresponds to type I dominance, then the dual solution corresponds to type II seesaw dominance and vice versa. The other solutions then in general correspond to a hybrid seesaw with type I and type II contributions to m_ν being of the same order. An important consequence of this result is that, *knowing only the values of the input parameters, one can determine if there are solutions with one seesaw type dominance, but cannot decide which particular seesaw type dominates.*

We have explored the behaviour of the analytic solutions for f in a number of interesting limiting cases: (a) one seesaw type dominates; (b) the two seesaw contributions almost

seesaw formula	neutrino Yukawa y	section	one gen.	two gen.	three gen.
eq. (1.3)	generic (no duality)	3.1 and 3.2	2	4	–
eq. (1.3)	symmetric	2.3	2	4	8
eq. (1.3)	antisymmetric	appendix A	1	2	2
eq. (C.3)	generic (no duality)	appendix C	2 or 0	–	–
eq. (C.3)	Hermitian	appendix C	2	4	8
eq. (C.3)	anti-Hermitian	appendix C	2 or 0	4 or 0	8 or 0

Table 1: Multiplicity of solutions for the Majorana-type Yukawa coupling matrix f. The first column indicates the seesaw equation under consideration, the second column specifies the assumption made on the structure of the Dirac-type Yukawa coupling matrix y, the third column indicates the section where the multiplicity was derived. Fourth, fifth and sixth columns give the number of solutions in the case of one, two and three lepton generations, respectively. The dash indicates the cases for which the number of solutions was not derived.

cancel each other; (c) the eigenvalues of y are strongly hierarchical; (d) one light neutrino mass vanishes. By analyzing several numerical examples, we found that the masses of RH neutrinos exhibit the following generic features: (i) differ strongly for different dual solutions for f; (ii) depend crucially on the light neutrino mass spectrum; (iii) depend weakly on variations of θ_{13} within its allowed range and on variations of the low-energy leptonic CP-violating phases; (iv) depend strongly on the hierarchy among the eigenvalues of y only in the cases with predominant type I seesaw contribution to m_ν; (v) may become quasi-degenerate in several regions of the parameter space which are close to the points where the level crossings occur; (vi) can be at TeV scale or lighter even when v_R is much larger and/or some entries of the neutrino Yukawa coupling matrix y are of order one.

We have also studied (in appendix A) the case of antisymmetric Dirac-type Yukawa coupling matrices y and showed that our linearization procedure works in that case as well, leading to simple analytic expressions for f. The multiplicity of solutions in that case was found to be 1, 2 and 2 for one, two and three lepton generations.

In addition, we considered (in appendix C) an alternative realization of the discrete LR symmetry, in which it acts as parity transformation rather than charge conjugation. This leads to type II and type I contributions to m_ν depending on f and f^* rather than on the same matrix f. In that case exactly the same duality of solutions holds, provided that the matrix y is Hermitian or anti-Hermitian and invertible. In the present paper we gave explicit formulas for the solutions for one and two lepton generations, and briefly outlined the approach to the three-generation case. The multiplicity of solutions for n lepton generations was shown to be 2^n in the case of Hermitian y, whereas for anti-Hermitian y there are either 2^n or no solutions. The latter possibility is a consequence of the non-analytic dependence of the seesaw relation on f in this realization of the discrete LR symmetry. Our results on the numbers of solutions for f in various cases are summarized in table 1.

Since one of our main goals was to provide a guidance for building specific models incorporating the LR-symmetric seesaw, we have discussed the structure of the seesaw formula in the minimal LR model, Pati-Salam model and in several SO(10) models with

renormalizable Yukawa couplings. Our analysis shows that in most cases the phenomenon of seesaw duality is realized: one can identify a matrix of couplings f which enters in the seesaw formula both directly and through its inverse. Such matrix can be reconstructed by making use of the methods developed in this paper.

We have discussed briefly the stability of our results with respect to the renormalization group running effects, which below the LR symmetry breaking scale may result in modifications of the relations $f_L = f_R$ and $y^T = \pm y$ that were at the basis of our analysis. Conditions were found under which these renormalization effects will not spoil our reconstruction of the matrix f from the input data. A comprehensive study of this issue, however, will require a dedicated effort.

Finally, we pointed out that the usual leptogenesis mechanism of the generation of baryon asymmetry of the universe produces an asymmetry that strongly depends on the adopted solution for f and therefore may in principle help discriminate between the eight allowed solutions. Moreover, the level crossing between the masses of the two lightest RH neutrinos, which occurs for a number of solutions, may provide a resonant enhancement of the produced asymmetry.

Acknowledgments

A. Reconstruction of f in the case of antisymmetric y

Consider the case of antisymmetric Dirac-type neutrino Yukawa coupling, $y = -y^T$. A crucial difference with respect to the case of symmetric y is the fact that $(2n+1)$-dimensional antisymmetric matrices are not invertible, so that the usual duality among the solutions for f does not hold for an odd number of lepton generations. In the case of one generation, one trivially has $y = 0$, and the seesaw is purely of type II.

The case of two generations is more interesting. An antisymmetric 2×2 matrix is defined by a single parameter. Consider the system (3.9) for $y_{\mu 2} = y_{\tau 3} = 0$ and $y_{\mu 3} = -y_{\tau 2} \equiv \bar{y}$. Using the same linearization procedure as in section 3.2, one easily finds the general solution:

$$ f = \frac{x\lambda}{x\lambda - \bar{y}^2} m \qquad \text{with} \qquad (x\lambda)^2 - \left(2\bar{y}^2 + x \det m\right) x\lambda + \bar{y}^4 = 0 . \qquad \text{(A.1)} $$

The solutions $\lambda_\pm$ of the quadratic equation correspond to a pair of dual solutions $f_\pm$. The duality relation (3.12) is replaced by $x^2 \lambda_+ \lambda_- = \bar{y}^4$. Notice that both solutions are proportional to the matrix m.

While in general the system (3.9) has four solutions, two of them are singular when y is exactly antisymmetric, as assumed above. This can be understood as follows: consider

the case where the diagonal entries of y are equal to zero and the off-diagonal ones satisfy $y_{\mu 3} \approx \bar{y}$ and $y_{\tau 2} \approx -\bar{y}$, so that y is nearly antisymmetric, Then f is still (approximately) given by eq. (A.1), while the quartic equation for λ reads

$$\left[(x\lambda)^2 - \left(2\bar{y}^2 + x \det m\right) x\lambda + \bar{y}^4\right] (x\lambda - \bar{y}^2)^2 \approx 0 \, , \tag{A.2}$$

to be compared with eq. (A.1). The pair of dual solutions corresponding to $x\lambda \to \bar{y}^2$ is not physical in the limit, since all the matrix elements of f diverge.

Consider now the relative size of type I and II seesaw contributions to m_ν. The two solutions of the characteristic equation for λ in eq. (A.1) can be written as

$$x\lambda_\pm = \frac{x \det m}{2} \left[1 + \frac{d}{2} \pm (1+d)^{1/2}\right] \, , \qquad d \equiv \frac{4\bar{y}^2}{x \det m} \, . \tag{A.3}$$

A straightforward calculation shows that both m_ν^{II} and m_ν^I are proportional to m_ν and their ratio is

$$r_\pm^{II/I} \equiv \left. \frac{(m_\nu^{II})_{ij}}{(m_\nu^I)_{ij}} \right|_\pm = -\frac{2}{d} \left[1 + \frac{d}{2} \pm (1+d)^{1/2}\right] \, . \tag{A.4}$$

For $|d| \ll 1$, $r_\pm^{II/I} \approx -(4/d)^{\pm 1}$, so that one seesaw type dominates. For $|d| \sim 1$, $r_\pm^{II/I} \sim 1$, corresponding to hybrid seesaw. For $|d| \gg 1$, $r_\pm^{II/I} \approx -1$, which implies a considerable cancellation of the two seesaw contributions. These three cases resemble closely the corresponding limits in the one lepton generation scenario (section 3.1).

Let us now turn to the realistic case of three lepton generations. As was pointed out above, the usual duality does not apply since y has one vanishing eigenvalue. However, a simple analytic reconstruction of the matrix f is still possible, as we show below.

Any antisymmetric 3×3 matrix can be written as

$$y = \begin{pmatrix} 0 & y_3 & y_2 \\ -y_3 & 0 & y_1 \\ -y_2 & -y_1 & 0 \end{pmatrix} = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -y_{123} \\ 0 & y_{123} & 0 \end{pmatrix} U^T \equiv U y' U^T \, , \tag{A.5}$$

where $y_{123} \equiv \sqrt{|y_1|^2 + |y_2|^2 + |y_3|^2}$ and the unitary matrix U is given by

$$U = \frac{1}{y_{123} y_{23}} \begin{pmatrix} y_1^* y_{23} & -y_{23}^2 & 0 \\ -y_2^* y_{23} & -y_2^* y_1 & y_3 y_{123} \\ y_3^* y_{23} & y_3^* y_1 & y_2 y_{123} \end{pmatrix} \, , \tag{A.6}$$

with $y_{23} \equiv \sqrt{|y_2|^2 + |y_3|^2}$. Defining $m' \equiv U^\dagger m U^*$ and $f' \equiv U^\dagger f U^*$, one can write the seesaw equation as

$$m' = f' + \frac{1}{x} y' f'^{-1} y' \, . \tag{A.7}$$

The entries of the first row of f' are uniquely determined as $f'_{1i} = m'_{1i}$, whereas to find the 2-3 block one needs to implement the usual linearization procedure. This yields

$$f'_{22} = \frac{x\lambda m'_{22} - y_{123}^2 m'^2_{12}}{x\lambda - y_{123}^2 m'_{11}} \, , \qquad f'_{23} = \frac{x\lambda m'_{23} - y_{123}^2 m'_{12} m'_{13}}{x\lambda - y_{123}^2 m'_{11}} \, , \qquad f'_{33} = \frac{x\lambda m'_{33} - y_{123}^2 m'^2_{13}}{x\lambda - y_{123}^2 m'_{11}} \, , \tag{A.8}$$

where λ is the solution of the quadratic equation

$$(x\lambda)^2 - (2y_{123}^2 m'_{11} + x \det m')x\lambda + y_{123}^4 m'^2_{11} = 0\,. \tag{A.9}$$

The two solutions $\lambda_{\pm}$ have the same form as in eq. (A.3), but with $d \equiv 4y_{123}^2 m'_{11}/(x \det m')$. Even though the usual duality property does not hold in this case, a *different duality* between the two solutions is present: one finds $x\lambda_{+}\lambda_{-} = y_{123}^4 m'^2_{11}$ and $f'_{+} + f'_{-} = \tilde{m}$, where $\tilde{m}_{ij} \equiv m'_{ij} + m'_{1i}m'_{1j}/m'_{11}$. The two solutions for f in the original basis are obtained as $f_{\pm} = U f'_{\pm} U^T$, where U is given in eq. (A.6).

Thus, for an antisymmetric y the realistic case of three lepton generations can be reduced to that of an effective two-generation system, so that there are only two solutions of the LR seesaw formula for f, which are related by a modified duality.

Note that the method of counting the solutions developed in section 2.3 does not apply to the case of antisymmetric y: for one and three generations, because y is not invertible, and for two generations because this is a degenerate case. Indeed, for $n = 2$ one finds $\tilde{f}^{-1} = [-\det y/(x \det f)] \cdot \tilde{f}$, so that the equation $\tilde{m} = \tilde{f} + \tilde{f}^{-1}$, which replaces eq. (2.10) in the case $y^T = -y$, is linear rather than quadratic in $\tilde{f}$. This equation results in a quadratic equation for $\det f$, leading to two solutions for f.

B. Relative size of type I and II seesaw in the presence of flavor mixing

In the simple case of one or more unmixed generations the relative size of type I and type II seesaw contribution to m_ν was discussed in section 3.1. Here we analyze this issue in the case of 2 and 3 generations with flavor mixing.

Due to the mixing, each entry of m_ν receives contributions from type I and II seesaw in different proportions. For simplicity, we will present only the conditions for the dominance of one seesaw type in all the matrix elements of m (extensions to more general cases can be easily obtained). In what follows, the relations involving $m_{\alpha\beta}$ and y_i will therefore be assumed to hold for each α, β and i, if not otherwise stated.

Consider first the two-generation case. From eq. (3.10) one can see that, when a solution λ_1 satisfies $|x\lambda_1| \gg |y_i y_j|$ $(i, j = 2, 3)$, one obtains $f_1 \simeq m$ (assuming that $m_{\mu\mu}$ and $m_{\tau\tau}$ are of the same order). This corresponds to the dominant type II seesaw. Then the dual solution $x\lambda_2 = x\hat{\lambda}_1 = y_2^2 y_3^2/(x\lambda_1)$ has modulus $\ll |y_i y_j|$ and the corresponding matrix $f_2 = \hat{f}_1$ takes the form obtained in type I seesaw case. In analogy with eq. (3.2), let us define the set of parameters

$$d^{ij}_{\alpha\beta\gamma\delta} \equiv \frac{v_L}{v_R}\frac{4v^2 y_i y_j}{(m_\nu)_{\alpha\beta}(m_\nu)_{\gamma\delta}} = \frac{4y_i y_j}{x m_{\alpha\beta}m_{\gamma\delta}}\,, \quad i,j = 2,3\,, \quad \alpha,\beta,\gamma,\delta = \mu,\tau\,. \tag{B.1}$$

Notice that eqs. (3.13) and (3.14) imply that a necessary condition to have $|x\lambda_1| \gg |y_i y_j|$ is $|m_{\alpha\beta}m_{\gamma\delta}| \gg 4|y_i y_j/x|$, that is,

$$|d^{ij}_{\alpha\beta\gamma\delta}| \ll 1\,. \tag{B.2}$$

Therefore, this limit ensures the existence of a pair of dual solutions $f_{1,2}$ with the dominance of one seesaw type, in analogy with the limit $|d| \ll 1$ in the one-generation case. The

solutions $\lambda_{1,2}$ are defined by choosing r_+ in eq. (3.13) and can be expanded in terms of the small parameters as

$$x\lambda_1 = x \det m \left[1 + \frac{k}{x(\det m)^2} + \frac{3(\det m)^2 y_2^2 y_3^2 - k^2}{x^2(\det m)^4} + \dots \right] ,$$

$$x\lambda_2 \equiv x\hat{\lambda}_1 = \frac{y_2^2 y_3^2}{x \det m} \left[1 - \frac{k}{x(\det m)^2} + \dots \right] .$$

(B.3)

The solutions for f are related by $f_1 = m - f_2$ with

$$f_2 \approx \frac{1}{x \det m} \begin{pmatrix} -y_2^2 \left[m_{\tau\tau} + \frac{y_3^2 m_{\mu\mu} \det m - m_{\tau\tau} k}{x(\det m)^2} \right] & y_2 y_3 m_{\mu\tau} \left[1 - \frac{y_2 y_3 \det m + k}{x(\det m)^2} \right] \\ \dots & -y_3^2 \left[m_{\mu\mu} + \frac{y_2^2 m_{\tau\tau} \det m - m_{\mu\mu} k}{x(\det m)^2} \right] \end{pmatrix} .$$

(B.4)

The second terms in the square brackets represent the leading order correction to the pure type I seesaw.

In general, condition (B.2) does not guarantee that the other dual pair of solutions $f_{3,4}$ corresponds to one seesaw type dominance. If no special cancellations occur, eq. (B.2) implies, in particular, $|\det m| \gg 4|y_i y_j/x|$. If this condition is satisfied, then f_3 and f_4 are of hybrid type. This novel feature with respect to the one-generation case is a genuine effect of flavor mixing. In fact, the solutions corresponding to the choice of r_- in eq. (3.13) are given, up to higher orders in small parameters, by

$$x\lambda_{3,4} = -\frac{k \pm s}{2 \det m} \left[1 \mp \frac{s}{x(\det m)^2} + \dots \right] , \qquad s \equiv \sqrt{k^2 - (2y_2 y_3 \det m)^2} .$$

(B.5)

Notice that $\lambda_{3,4}$ are dual to each other. The corresponding solutions for f (to leading order) are

$$f_{3,4} \approx \frac{1}{2s} \begin{pmatrix} m_{\mu\mu}(s \pm k) \mp 2m_{\tau\tau} y_2^2 \det m & m_{\mu\tau}(s \pm k \pm 2y_2 y_3 \det m) \\ \dots & m_{\tau\tau}(s \pm k) \mp 2m_{\mu\mu} y_3^2 \det m \end{pmatrix} .$$

(B.6)

Roughly, $(f_{3,4})_{ij} \sim m_{\alpha\beta}$ and $F_{3,4} = \lambda_{3,4} \sim y_i y_j/x$. Therefore, type II and type I contributions to m_ν are of the same order: $m_\nu^{II} \sim v_L m_{\alpha\beta}$ and $m_\nu^I \sim v^2 y_i y_j f_{kl}/(v_R F) \sim v_L m_{\alpha\beta}$; this proves that the solutions $f_{3,4}$ are of hybrid type.

If a special cancellation in $\det m$ occurs, also the solutions $f_{3,4}$ are dominated by one type of seesaw. Indeed, in the limit $\det m \to 0$ eq. (3.13) becomes

$$x\lambda_{1,3} \approx \pm \frac{1}{2} \left(\sqrt{kx + 4y_2^2 y_3^2} + \sqrt{kx} \right) , \qquad x\lambda_{2,4} \equiv x\hat{\lambda}_{1,3} \approx \pm \frac{1}{2} \left(\sqrt{kx + 4y_2^2 y_3^2} - \sqrt{kx} \right) ,$$

(B.7)

where $k \approx (m_{\mu\mu} y_3 + m_{\tau\tau} y_2)^2$. When eq. (B.2) holds, one has $|kx| \gg 4y_2^2 y_3^2$. As a consequence, $|x\lambda_1| = |x\lambda_3| \gg |y_i y_j|$, so that $f_{1,3}$ are solutions with dominant type II seesaw. By duality, $f_{2,4}$ correspond to the dominant type I seesaw.

Note that a suppression in $\det m_\nu$ is phenomenologically motivated if the neutrino spectrum has normal mass hierarchy. In this case the full 3×3 mass matrix m_ν is dominated

by large entries in the 2×2 $\mu\tau$-block. This block has to incorporate the maximal mixing observed in the atmospheric neutrino oscillations as well as the hierarchy $\Delta m_{\text{sol}}^2 \ll \Delta m_{\text{atm}}^2$. These two requirements imply that the determinant of the $\mu\tau$-block is suppressed.

When one abandons the condition (B.2), that is, when at least one $|d_{\alpha\beta\gamma\delta}^{ij}| \gtrsim 1$, all four solutions are of hybrid type. In other words, *some sets of input parameters necessarily imply a hybrid seesaw scenario.*

In the limit when all $|d_{\alpha\beta\gamma\delta}^{ij}| \gg 1$, one has $x\lambda_i \approx r_\pm/4 \approx \pm y_2 y_3$, and eq. (3.10) implies that (at least the diagonal) matrix elements of f grow as $1/(|x\lambda| - |y_2 y_3|)$, thus ending up to be much larger than $m_{\alpha\beta}$. This means that one enters the cancellation region, where type I and II contributions are almost equal in absolute value and opposite in sign. In this limit, the matrix elements of f will violate perturbative unitarity. This means that for some values of the input parameters, there are no physically acceptable solutions for f. In other words, *the LR symmetric seesaw mechanism cannot reproduce certain sets of the input parameters and thus, in principle, can be ruled out.*

Consider now the three-generation case. In analogy with eq. (B.1), one finds that a necessary condition for the dominance of one seesaw type is

$$
|d_{\alpha\beta\gamma\delta}^{ij}| \equiv \frac{v_L}{v_R} \left| \frac{4v^2 y_i y_j}{(m_\nu)_{\alpha\beta}(m_\nu)_{\gamma\delta}} \right| = \left| \frac{4 y_i y_j}{x m_{\alpha\beta} m_{\gamma\delta}} \right| \ll 1, \quad i,j = 1,2,3, \quad \alpha,\beta,\gamma,\delta = e,\mu,\tau.
$$
(B.8)

In this case, the solutions λ_i of eq. (3.29) can be expanded in the small parameters $d_{\alpha\beta\gamma\delta}^{ij}$, which we denote collectively by d. The largest solution turns out to be

$$
\lambda_1 = \det m \left[1 + \frac{A}{(\det m)^2} + \frac{3Y^2 S(\det m)^2 - A^2}{(\det m)^4} + \mathcal{O}(d^3) \right].
$$
(B.9)

Noting that $Y^2/\lambda_1^2 \sim d^3$, it is straightforward to verify with eq. (3.25) that f_1 corresponds to the dominant type II seesaw. As a consequence, the three RH neutrino masses satisfy $M_{1,2,3} \gg y_i y_j M_s$, where M_s is the seesaw scale defined as $M_s \equiv v^2/m_3$ (normal mass ordering) of v^2/m_2 (inverted mass ordering).

Three more solutions are given by $\lambda_{2,3,4} = \bar{\lambda}_{2,3,4}[1 + \mathcal{O}(d)]$, where $\bar{\lambda}_{2,3,4}$ are the roots of the cubic equation

$$
\bar{\lambda}^3 + \frac{A}{\det m} \bar{\lambda}^2 + Y^2 S \bar{\lambda} + Y^2 \det m = 0.
$$
(B.10)

It is easy to see that $\lambda_{2,3,4}/\lambda_1 \sim d$. From eq. (3.25) one finds

$$
(f_k)_{ij} \approx \frac{(\bar{\lambda}_k^3 - Y^2 \det m) m_{ij} + \bar{\lambda}_k^2 A_{ij} - Y^2 \bar{\lambda}_k S_{ij}}{\bar{\lambda}_k^3 - Y^2 \bar{\lambda}_k S - 2Y^2 \det m}, \quad k = 2,3,4.
$$
(B.11)

One can verify that for these three solutions type I and II contributions to m_ν are of the same order (hybrid seesaw). Since $\lambda_k = \det f_k$ are suppressed with respect to λ_1, only two RH neutrinos are have masses that are larger than the seesaw scale, $M_{2,3} \gg y_i y_j M_s$, while $M_1 \sim y_i y_j M_s$.

The dual solutions, $\hat{\lambda}_{2,3,4} = -Y^2/\lambda_{2,3,4}$, are suppressed by a factor $\sim d^2$ with respect to λ_1. The corresponding structures for f are

$$(\hat{f}_k)_{ij} \approx \frac{-Y^2(\bar{\lambda}_k S + \det m)m_{ij} - \bar{\lambda}_k^2 A_{ij} + Y^2\bar{\lambda}_k S_{ij}}{\bar{\lambda}_k^3 - Y^2\bar{\lambda}_k S - 2Y^2\det m}, \quad k = 2,3,4. \tag{B.12}$$

Note that $f_k + \hat{f}_k = m$, as required by duality. The solutions $\hat{f}_k$ also correspond to type I and II contributions to m_ν being of the same order (hybrid seesaw). The masses of the RH neutrinos satisfy $M_3 \gg y_i y_j M_s$, $M_{1,2} \sim y_i y_j M_s$.

Finally, the solution dual to λ_1 is given by

$$\hat{\lambda}_1 = -\frac{Y^2}{\det m}\left[1 - \frac{A}{(\det m)^2} + \mathcal{O}(d^2)\right]. \tag{B.13}$$

This is the smallest solution, since $\hat{\lambda}_1/\lambda_1 \sim d^3$. By duality, it corresponds to the dominant type I seesaw, as one can check explicitly expanding eq. (3.25). All three RH neutrino masses are approximately at the seesaw scale, $M_{1,2,3} \sim y_i y_j M_s$.

As already mentioned, the parameters $d^{ij}_{\alpha\beta\gamma\delta}$ may not all be small at the same time. In that case, for any of the eight solutions f, only some of the elements of the matrix m_ν may be dominated by one seesaw type, while one or more other elements receive significant contributions from both seesaw types.

A remark is in order on the special case when $\det m$ is suppressed. For example, one may have $|\det m| \ll |Y|$ even if eq. (B.8) holds. Clearly, in the limit $\det m \to 0$ the expansions (B.9)–(B.13) do not apply. We showed in section 3.3.2 that, when $\det m = 0$, two pairs of dual solutions for λ are equal in absolute value and opposite in sign to the other two dual pairs. Moreover, the equation for λ has the same form as in the two-generation case (with the redefinitions (3.37)). In particular, the relative size of type I and II seesaw contributions to m_ν may be evaluated as we did above for the two-generation case: when eq. (B.8) holds, there are two opposite solutions $\pm\lambda_1$ which lead to the dominant type II seesaw, while their duals correspond to the dominant type I seesaw. The other two opposite pairs of dual solutions correspond in general to hybrid seesaw, unless a special suppression in $\sqrt{A}$ occurs.

C. Alternative realization of the discrete left-right symmetry

Consider a different realization of the discrete LR symmetry in the minimal LR model, in which it acts as parity transformation. Eq. (5.6) is then replaced by

$$L \leftrightarrow (L^c)^c \equiv \begin{pmatrix} N_R \\ l_R \end{pmatrix}, \qquad \Phi \leftrightarrow \Phi^\dagger, \qquad \Delta_L \leftrightarrow \Delta_R^*. \tag{C.1}$$

This possibility is almost as popular in the literature (see, e.g., [36, 24]) as the one we adopted before, even though it does not allow an SO(10) embedding of the minimal LR model. Eqs. (C.1) and (5.1) imply

$$f \equiv f_L = f_R^*, \qquad g = g^\dagger, \qquad h = h^\dagger. \tag{C.2}$$

In this case the seesaw relation is

$$m_\nu \simeq v_L f - \frac{v^2}{v_R} y (f^*)^{-1} y^T \,, \tag{C.3}$$

where y is defined in eq. (5.3). The difference in complex phases between the seesaw formulas (1.3) and (C.3) can, in principle, lead to a different phenomenology if CP is violated in the leptonic sector.

It can be easily checked by direct substitution into eq. (C.3) that, if the matrix y is Hermitian or anti-Hermitian and invertible, this equation possesses the same duality property as eq. (1.3): namely if a matrix f solves eq. (C.3), so does its dual $\hat{f} \equiv m_\nu/v_L - f$. The above conditions on y are both necessary and sufficient for the duality to hold.

Eqs. (C.2) and (5.3) imply that y is actually Hermitian if $h = 0$ (which holds, e.g., in the SUSY case) or if the two VEVs of the bidoublet v_1 and v_2 are both real (this is not the case in general, since both VEVs can be non-zero, and their relative phase cannot be rotated away [36]). If y is Hermitian, one can diagonalize it according to $y = U y_d U^\dagger$ with U unitary, and go to the basis where y is diagonal and real by absorbing U in the definition of m_ν and f (analogously to eq. (2.8)).

Consider first the one-generation case with an arbitrary complex y. Defining $f \equiv |f|e^{i\phi}$, one can obtain from eq. (C.3)

$$e^{i\phi} = \frac{m|f|}{|f|^2 - y^2/x} \,. \tag{C.4}$$

The value of $|f|$ is determined by the requirement that that numerator and denominator on the right hand side have the same moduli (recall that in our conventions all the VEVs are real and positive, and so is x):

$$|f|^4 - \left[|m|^2 + \frac{2\cos\chi|y|^2}{x} \right] |f|^2 + \frac{|y|^4}{x^2} = 0 \,. \tag{C.5}$$

Here we used $y^2 \equiv |y|^2 e^{i\chi}$. Eq. (C.5) has (two) real and positive solutions for $|f|^2$ if and only if

$$|m|^2 > \frac{2|y|^2}{x}(1 - \cos\chi) \,. \tag{C.6}$$

This means that, if the input parameters do not obey this inequality, no value of f satisfies the seesaw formula (C.3). In other words, such a set of input parameters cannot be explained by the LR symmetric seesaw mechanism. This problem never occurs in the case of eq. (1.3), which is analytic in the elements of f and thus always has solutions.

The requirement that y be Hermitian or anti-Hermitian reduces in the one generation case to the condition that y be real ($\chi = 0$) or purely imaginary ($\chi = \pi$). Let us focus on the $\chi = 0$ case. Eq. (C.6) is then always satisfied and the two solutions of eq. (C.5) are

$$|f_\pm| = \frac{|m|}{2} \left[(1 + |d|)^{1/2} \pm 1 \right] \,, \tag{C.7}$$

where d was defined in eq. (3.2). The discussion of the dependence of $|f_\pm|$ on $|d|$ in different limiting cases is analogous to that in section 3.1. Finally, the phases $\phi_\pm \equiv \arg f_\pm$

are determined uniquely by plugging eq. (C.7) into eq. (C.4). Thus, if y is real, eq. (C.3) has two solutions.

Consider now two lepton generations, limiting ourselves to the case $y = y^\dagger$ and working in the basis where y is diagonal and real. The linearization procedure is the same as in section 3.2. However, to close the system of equations one has to consider both eq. (C.3) and its complex conjugate, so that the linearized system contains 6 equations. Defining $\lambda = |\lambda|e^{i\rho}$, one arrives at the solution which is analogous to that in eq. (3.10):

$$
f = \frac{x|\lambda|}{(x|\lambda|)^2 - y_2^2 y_3^2} \begin{pmatrix} x|\lambda|m_{\mu\mu} + y_2^2 m_{\tau\tau}^* e^{i\rho} & x|\lambda|m_{\mu\tau} - y_2 y_3 m_{\mu\tau}^* e^{i\rho} \\ \dots & x|\lambda|m_{\tau\tau} + y_3^2 m_{\mu\mu}^* e^{i\rho} \end{pmatrix}. \tag{C.8}
$$

The parameter λ is determined, as usual, from the equation $\lambda = \det f(\lambda)$, which reads:

$$
\begin{aligned}
& \left(x^2|\lambda|^2 - y_2^2 y_3^2\right)^2 - x^3|\lambda|^2(|m_{\mu\mu}|^2 y_3^2 + 2|m_{\mu\tau}|^2 y_2 y_3 + |m_{\tau\tau}|^2 y_2^2) \\
& -x^2|\lambda| \left[\det m e^{-i\rho} x^2|\lambda|^2 + (\det m)^* e^{i\rho} y_2^2 y_3^2\right] = 0.
\end{aligned} \tag{C.9}
$$

The imaginary part of this equation implies $\rho \equiv \arg \lambda = \arg \det m + l\pi$ with $l = 0, 1$ (barring the special cases $\det m = 0$ and $x|\lambda| = y_2 y_3$). Then $|\lambda|$ satisfies a quartic equation, which is completely analogous to that in eq. (3.11). Therefore, its four solutions are easily found analytically and one can verify that for each $l = 0$ and $l = 1$ there is only one pair of dual solutions which yield real and positive $|\lambda|$ and therefore are acceptable. As a consequence, also in this scenario there are four solutions for the matrix f.

The three-generation case can be considered quite analogously. The linearization procedure yields a closed set of 12 linear equations for the elements of the matrices f and f^*, and solving the characteristic equation one arrives at eight possible solutions for f.

It is actually not difficult to show that the number of solutions for n lepton generations is always 2^n, provided that the matrix y is Hermitian and invertible. Indeed, going into the basis where y is diagonal an real and performing the same transformations as in section 2.3, one arrives at the equation

$$
\tilde{m} = \tilde{f} - (\tilde{f}^*)^{-1}, \tag{C.10}
$$

where $\tilde{m}$ and $\tilde{f}$ are symmetric. The matrix $\tilde{f}$ can be diagonalized as $\tilde{f} = V\tilde{f}_d V^T$, where V is unitary and $\tilde{f}_d$ diagonal and real; it is easy to see that then $(\tilde{f}^*)^{-1}$ is diagonalized by the same transformation. Therefore, multiplying eq. (C.10) by $V^\dagger$ on the left and by V^* on the right, one diagonalizes its right hand side, and so also the left hand side. Thus, V is a unitary matrix that diagonalizes the matrix $\tilde{m}$ and therefore is determined by the known quantities m and y. In the diagonal basis eq. (C.10) yields

$$
\tilde{f}_{di}^2 - \tilde{m}_{di}\tilde{f}_{di} - 1 = 0, \qquad i = 1, \dots n. \tag{C.11}
$$

This gives two values of $\tilde{f}_{di}$ for each i:

$$
\tilde{f}_{di} = \frac{\tilde{m}_{di}}{2} \pm \sqrt{\frac{\tilde{m}_{di}^2}{4} + 1}. \tag{C.12}
$$

A full n-generation solution is obtained by picking one of these two values for each i, which yields 2^n solutions for the matrix $\tilde{f}$ and therefore for f.

The same method can also be applied to the case of anti-Hermitian y (making use of the fact that iy is Hermitian). Eq. (C.12) is then replaced by

$$\tilde{f}_{di} = \frac{\tilde{m}_{di}}{2} \pm \sqrt{\frac{\tilde{m}_{di}^2}{4} - 1} \,, \tag{C.13}$$

This leads to 2^n solutions for f, provided that all $\tilde{m}_{di}$ satisfy $\tilde{m}_{di}^2 > 4$; otherwise the seesaw equation (C.3) has no solutions.

A procedure similar to that described below eq. (C.10) (but making use of a complex orthogonal transformation rather than of a unitary one) was first used in [13] in order to solve eq. (1.3) with symmetric y. In both cases, this approach allows not only to count the solutions of the seesaw equation, but also to obtain them explicitly.

References

[1] P. Minkowski, $\mu \to e\gamma$ at a rate of one out of 1-billion muon decays?, Phys. Lett. **B 67** (1977) 421;
M. Gell-Mann, P. Ramond, and R. Slansky, *Complex spinors and unified theories*, Print-80-0576 (CERN), in Supergravity, P. van Nieuwenhuizen and D.Z. Freedman ed., North Holland, Amsterdam 1979, p.315;
T. Yanagida, *Horizontal gauge symmetry and masses of neutrinos*, in proc. of the *Workshop on the Unified Theory and Baryon Number in the Universe*, O. Sawada and A. Sugamoto eds., KEK report 79-18, 1979, p.95, Tsukuba, Japan;
S.L. Glashow, *The future of elementary particle physics*, in Cargese 1979, Proceedings, *Quarks and Leptons*, 687-713 and Harvard Univ. Cambridge, HUTP-79-A059 (1979) 40p;
R.N. Mohapatra and G. Senjanovic, *Neutrino mass and spontaneous parity nonconservation*, Phys. Rev. Lett. **44** (1980) 912.

[2] M. Magg and C. Wetterich, *Neutrino mass problem and gauge hierarchy*, Phys. Lett. **B 94** (1980) 61;
G. Lazarides, Q. Shafi and C. Wetterich, *Proton lifetime and fermion masses in an so(10) model*, Nucl. Phys. **B 181** (1981) 287;
J. Schechter and J.W.F. Valle, *Neutrino masses in* SU(2) × U(1) *theories*, Phys. Rev. **D 22** (1980) 2227;
R.N. Mohapatra and G. Senjanovic, *Neutrino masses and mixings in gauge models with spontaneous parity violation*, Phys. Rev. **D 23** (1981) 165;
E. Ma and U. Sarkar, *Neutrino masses and leptogenesis with heavy Higgs triplets*, Phys. Rev. Lett. **80** (1998) 5716 [`hep-ph/9802445`].

[3] R. Foot, H. Lew, X.G. He and G.C. Joshi, *Seesaw neutrino masses induced by a triplet of leptons*, Z. Physik **C 44** (1989) 441;
E. Ma, *Pathways to naturally small neutrino masses*, Phys. Rev. Lett. **81** (1998) 1171 [`hep-ph/9805219`].

[4] M. Fukugita and T. Yanagida, *Baryogenesis without grand unification*, Phys. Lett. **B 174** (1986) 45.

[5] B. Dutta, Y. Mimura and R.N. Mohapatra, *CKM CP-violation in a minimal* SO(10) *model for neutrinos and its implications*, Phys. Rev. **D 69** (2004) 115014 [`hep-ph/0402113`];

B. Bajc, G. Senjanovic and F. Vissani, *Probing the nature of the seesaw in renormalizable* SO(10), *Phys. Rev.* **D 70** (2004) 093002 [hep-ph/0402140];

H.S. Goh, R.N. Mohapatra and S. Nasri, SO(10) *symmetry breaking and type-II seesaw,* *Phys. Rev.* **D 70** (2004) 075022 [hep-ph/0408139];

K.S. Babu and C. Macesanu, *Neutrino masses and mixings in a minimal* SO(10) *model,* *Phys. Rev.* **D 72** (2005) 115003 [hep-ph/0505200];

C.S. Aulakh and S.K. Garg, *MSGUT: from bloom to doom, Nucl. Phys.* **B 757** (2006) 47 [hep-ph/0512224].

[6] S. Bertolini, T. Schwetz and M. Malinsky, *Fermion masses and mixings in* SO(10) *models and the neutrino challenge to SUSY GUTs, Phys. Rev.* **D 73** (2006) 115012 [hep-ph/0605006].

[7] K. Matsuda, T. Fukuyama and H. Nishiura, SO(10) *GUT and quark-lepton mass matrices,* *Phys. Rev.* **D 61** (2000) 053001 [hep-ph/9906433].

[8] J.C. Pati and A. Salam, *Lepton number as the fourth color, Phys. Rev.* **D 10** (1974) 275.

[9] R.N. Mohapatra and J.C. Pati, *Left-right gauge symmetry and an 'isoconjugate' model of CP-violation, Phys. Rev.* **D 11** (1975) 566;

G. Senjanovic and R.N. Mohapatra, *Exact left-right symmetry and spontaneous violation of parity, Phys. Rev.* **D 12** (1975) 1502.

[10] H. Georgi, *The state of the art - gauge theories (talk), AIP Conf. Proc.* **23** (1975) 575;

H. Fritzsch and P. Minkowski, *Unified interactions of leptons and hadrons, Ann. Phys. (NY)* **93** (1975) 193.

[11] C.S. Aulakh, A. Melfo, A. Rasin and G. Senjanovic, *See-saw and supersymmetry or exact R-parity, Phys. Lett.* **B 459** (1999) 557 [hep-ph/9902409].

[12] E.K. Akhmedov and M. Frigerio, *Seesaw duality, Phys. Rev. Lett.* **96** (2006) 061802 [hep-ph/0509299].

[13] P. Hosteins, S. Lavignac and C.A. Savoy, *Quark-lepton unification and eight-fold ambiguity in the left-right symmetric seesaw mechanism, Nucl. Phys.* **B 755** (2006) 137 [hep-ph/0606078].

[14] M. Maltoni, T. Schwetz, M.A. Tortola and J.W.F. Valle, *Status of global fits to neutrino oscillations, New J. Phys.* **6** (2004) 122 [hep-ph/0405172].

[15] A. Strumia and F. Vissani, *Implications of neutrino data circa 2005, Nucl. Phys.* **B 726** (2005) 294 [hep-ph/0503246];

G.L. Fogli, E. Lisi, A. Marrone and A. Palazzo, *Global analysis of three-flavor neutrino masses and mixings, Prog. Part. Nucl. Phys.* **57** (2006) 742 [hep-ph/0506083].

[16] A. Goobar, S. Hannestad, E. Mortsell and H. Tu, *A new bound on the neutrino mass from the SDSS baryon acoustic peak, JCAP* **06** (2006) 019 [astro-ph/0602155];

J. Lesgourgues and S. Pastor, *Massive neutrinos and cosmology, Phys. Rept.* **429** (2006) 307 [astro-ph/0603494];

U. Seljak, A. Slosar and P. McDonald, *Cosmological parameters from combining the Lyman-alpha forest with CMB, galaxy clustering and sn constraints, JCAP* **10** (2006) 014 [astro-ph/0604335];

M. Cirelli and A. Strumia, *Cosmology of neutrinos and extra light particles after WMAP3,* *JCAP* **12** (2006) 013 [astro-ph/0607086];

G.L. Fogli et al., *Observables sensitive to absolute neutrino masses: a reappraisal after WMAP-3y and first minos results,* hep-ph/0608060.

[17] PARTICLE DATA GROUP Collaboration, W.M. Yao et al., *Review of particle physics*, *J. Phys.* **G 33** (2006) 1.

[18] W. Grimus and L. Lavoura, *Softly broken lepton numbers and maximal neutrino mixing*, *JHEP* **07** (2001) 045 [`hep-ph/0105212`].

[19] S. Lavignac, I. Masina and C.A. Savoy, $\tau \to \mu\gamma$ *and* $\mu \to e\gamma$ *as probes of neutrino mass models*, *Phys. Lett.* **B 520** (2001) 269 [`hep-ph/0106245`].

[20] A.S. Joshipura, E.A. Paschos and W. Rodejohann, *A simple connection between neutrino oscillation and leptogenesis*, *JHEP* **08** (2001) 029 [`hep-ph/0105175`].

[21] F.R. Gantmacher, *The theory of matrices*, vol. I. New York, Chelsea Pub. Co., 1959.

[22] A.S. Joshipura, E.A. Paschos and W. Rodejohann, *Leptogenesis in left-right symmetric theories*, *Nucl. Phys.* **B 611** (2001) 227 [`hep-ph/0104228`].

[23] E.K. Akhmedov, M. Frigerio and A.Y. Smirnov, *Probing the seesaw mechanism with neutrino data and leptogenesis*, *JHEP* **09** (2003) 021 [`hep-ph/0305322`].

[24] K. Kiers, M. Assis, D. Simons, A.A. Petrov and A. Soni, *Neutrinos in a left-right model with a horizontal symmetry*, *Phys. Rev.* **D 73** (2006) 033009 [`hep-ph/0510274`].

[25] C.S. Aulakh, *Fermion mass hierarchy in the Nu MSGUT. I: the real core*, `hep-ph/0602132`.

[26] L. Lavoura, H. Kuhbock and W. Grimus, *Charged-fermion masses in* $|SO(10)$: *analysis with scalars in 10+120*, *Nucl. Phys.* **B 754** (2006) 1 [`hep-ph/0603259`].

[27] W. Grimus and H. Kuhbock, *Fermion masses and mixings in a renormalizable* $SO(10) \times Z(2)$ *GUT*, `hep-ph/0607197`;
C.S. Aulakh, *MSGUT reborn?*, `hep-ph/0607252`.

[28] D. Chang, R.N. Mohapatra and M.K. Parida, *Decoupling parity and* $SU(2)_R$ *breaking scales: a new approach to left-right symmetric models*, *Phys. Rev. Lett.* **52** (1984) 1072.

[29] S. Antusch, J. Kersten, M. Lindner and M. Ratz, *Neutrino mass matrix running for non-degenerate see-saw scales*, *Phys. Lett.* **B 538** (2002) 87 [`hep-ph/0203233`];
S. Antusch, J. Kersten, M. Lindner, M. Ratz and M.A. Schmidt, *Running neutrino mass parameters in see-saw scenarios*, *JHEP* **03** (2005) 024 [`hep-ph/0501272`];
S.F. King and N.N. Singh, *Renormalisation group analysis of single right-handed neutrino dominance*, *Nucl. Phys.* **B 591** (2000) 3 [`hep-ph/0006229`].

[30] T. Hambye and G. Senjanovic, *Consequences of triplet seesaw for leptogenesis*, *Phys. Lett.* **B 582** (2004) 73 [`hep-ph/0307237`];
S. Antusch and S.F. King, *Type II leptogenesis and the neutrino mass scale*, *Phys. Lett.* **B 597** (2004) 199 [`hep-ph/0405093`].

[31] S. Antusch and S.F. King, *Leptogenesis in unified theories with type-II see-saw*, *JHEP* **01** (2006) 117 [`hep-ph/0507333`];
K.S. Babu, A. Bachri and H. Aissaoui, *Leptogenesis in minimal left-right symmetric models*, *Nucl. Phys.* **B 738** (2006) 76 [`hep-ph/0509091`];
N. Sahu and U. Sarkar, *Leptogenesis bound on neutrino masses in left-right symmetric models with spontaneous D-parity violation*, *Phys. Rev.* **D 74** (2006) 093002 [`hep-ph/0605007`].

[32] P.-H. Gu, H. Zhang and S. Zhou, *A minimal type-II seesaw model*, *Phys. Rev.* **D 74** (2006) 076002 [`hep-ph/0606302`].

[33] T. Hambye, M. Raidal and A. Strumia, *Efficiency and maximal CP-asymmetry of scalar triplet leptogenesis*, *Phys. Lett.* **B 632** (2006) 667 [`hep-ph/0510008`].

[34] M. Frigerio, T. Hambye and E. Ma, *Right-handed sector leptogenesis*, *JCAP* **09** (2006) 009 [`hep-ph/0603123`].

[35] J. Liu and G. Segre, *Reexamination of generation of baryon and lepton number asymmetries by heavy particle decay*, *Phys. Rev.* **D 48** (1993) 4609 [`hep-ph/9304241`];
M. Flanz, E.A. Paschos and U. Sarkar, *Baryogenesis from a lepton asymmetric universe*, *Phys. Lett.* **B 345** (1995) 248 [`hep-ph/9411366`];
L. Covi, E. Roulet and F. Vissani, *CP-violating decays in leptogenesis scenarios*, *Phys. Lett.* **B 384** (1996) 169 [`hep-ph/9605319`];
A. Pilaftsis, *CP-violation and baryogenesis due to heavy majorana neutrinos*, *Phys. Rev.* **D 56** (1997) 5431 [`hep-ph/9707235`];
A. Pilaftsis and T.E.J. Underwood, *Resonant leptogenesis*, *Nucl. Phys.* **B 692** (2004) 303 [`hep-ph/0309342`].

[36] N.G. Deshpande, J.F. Gunion, B. Kayser and F.I. Olness, *Left-right symmetric electroweak models with triplet Higgs*, *Phys. Rev.* **D 44** (1991) 837.